A general construction for unipolar and bipolar interpolative aggregation

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Abstract

We present a general view of interpolative aggregation, using the geometric realization of a poset, and its natural triangulation. This includes as particular cases the Choquet integral for capacities and $k$-ary capacities. In a second part, we address the problem of extending a given interpolative aggregation function, considered as unipolar, to bipolar aggregation.

Keywords: interpolation, triangulation, lattice, bipolar structure

1 Introduction

It is known from Lovász [6] that it is possible to consider the Choquet integral as a linear extension on $[0,1]^n$ of pseudo-Boolean functions (defined on $\{0,1\}^n$) (see Marichal [7]), and that this can be viewed as a linear interpolation between vertices of the hypercube $[0,1]^n$ (see Grabisch [1]). The argument behind this is to use what is called the canonical or natural triangulation of the hypercube, corresponding to simplices defined by permutations on $\{1,\ldots,n\}$.

In [3], Grabisch and Labreuche generalized this result by introducing the Choquet integral for $k$-ary capacities, just considering an adequate symmetrization of the classical Choquet integral.

This construction can be made fairly general, as we will show in this paper. One can define an interpolative aggregation function as a function performing an interpolation between combinations of particular predefined levels, assuming only a partial order between these predefined levels.

A second topic of this paper is how to symmetrize a given interpolative aggregation function in order to get the bipolar extension of it. For example, the Choquet integral w.r.t. bicapacities can be seen as the bipolar extension of the usual Choquet integral. This situation happens whenever the set of predefined levels contains a central level, considered as neutral, and levels greater (resp. smaller) than this central level are felt as “good” or positive, satisfactory (resp. bad, negative, unsatisfactory) by the decision maker (for a detailed motivation of bipolar scales rooted in psychology and its application to bicapacities, see [4]).

We present a general way of symmetrizing a given ordered structure (which should be at least an inf-semilattice) to get the bipolar extension of it, as well as the bipolar extension of the aggregation function.

2 Mathematical background

A lattice is a set $L$ endowed with a partial order $\leq$ such that for any $x, y \in L$ their least upper bound $x \lor y$ and greatest lower bound $x \land y$ always exist. For finite lattices, the greatest element of $L$ (denoted $\top$) and least element $\bot$ always exist. $x$ covers $y$ (denoted $x > y$) if $x > y$ and there is no $z$ such that $x > z > y$. The lattice is distributive if $\lor, \land$ obey distributivity. An element $j \in L$ is join-irreducible if it cannot be expressed as a supremum of other elements. Equivalently, $j$ is join-irreducible if it covers only one element. Join-irreducible elements covering $\bot$ are called atoms, and the lattice is atomistic if all join-irreducible elements are atoms. The set of all join-irreducible
elements of $L$ is denoted $\mathcal{J}(L)$.

For any $x \in L$, we say that $x$ has a complement in $L$ if there exists $x' \in L$ such that $x \wedge x' = \perp$ and $x \vee x' = \top$. The complement is unique if the lattice is distributive.

An important property is that in a distributive lattice, any element $x$ can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the minimal decomposition of $x$). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of $x$, and we denote by $\eta(x)$ the normal decomposition of $x$, defined as the set of join-irreducible elements smaller or equal to $x$, i.e., $\eta(x) := \{ j \in \mathcal{J}(L) \mid j \leq x \}$. Hence $\eta^*(x) \subseteq \eta(x)$, and

$$x = \bigvee_{j \in \eta^*(x)} j = \bigvee_{j \in \eta(x)} j.$$

Let us rephrase differently the above result. We say that $Q \subseteq L$ is a downset of $L$ if $x \in Q$, $y \in L$ and $y \leq x$ imply $y \in Q$. For any subset $P$ of $L$, we denote by $\mathcal{O}(P)$ the set of all downsets of $P$. Then the mapping $\eta$ is an isomorphism of $L$ onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff’s theorem).

We introduce now the notion of geometric realization of a lattice, following Koshevoy [5]. A first fact to notice is that if we consider for some partially ordered set $P$ the set $\mathcal{D}(P)$ of nonincreasing mappings from $P$ to $\{0, 1\}$, then Birkhoff’s theorem can be rephrased as follows: any distributive lattice $L$ is isomorphic to $\mathcal{D}(\mathcal{J}(L))$.

Consider next for any partially ordered set $P$ the set $\mathcal{C}(P)$ of nonincreasing mappings from $P$ to $[0, 1]$. It can be easily shown that $\mathcal{C}(P)$ is a convex polyhedron, whose set of vertices is $\mathcal{D}(P)$. We define the geometric realization of a distributive lattice $L$ by the set $\mathcal{C}(\mathcal{J}(L))$.

Example: If $L$ is a Boolean lattice $2^N$, with $N := \{1, \ldots, n\}$, then $\mathcal{J}(L) = N$ (atoms). We have $\mathcal{D}(\mathcal{J}(L)) = \{x : N \to \{0, 1\}, x \text{ nonincreasing}\}$, but since $N$ is an antichain, there is no restriction on $x$ and $\mathcal{D}(\mathcal{J}(L)) = \{0, 1\}^N$, i.e., it is the set of vertices of $[0, 1]^n$. Similarly, $\mathcal{C}(\mathcal{J}(L)) = [0, 1]^N$, which is the hypercube itself. □

Let us now introduce the natural triangulation of $\mathcal{C}(\mathcal{J}(L))$, following Koshevoy again. It consists in partitioning $\mathcal{C}(\mathcal{J}(L))$ into simplices whose vertices are in $\mathcal{D}(\mathcal{J}(L))$. To each chain, say $C := \{x_0 < x_1 < \cdots < x_p\}$, in $\mathcal{D}(\mathcal{J}(L))$ corresponds a $n$-dimensional simplex $\sigma(C) := \text{co}(x_0, x_1, \ldots, x_p)$. It can be shown that these simplices cover $\mathcal{C}(\mathcal{J}(L))$ such that any $f$ in $\mathcal{C}(\mathcal{J}(L))$ belongs to the interior of a unique simplex. Any $f$ in $\sigma(C)$ writes

$$f = \sum_{i=0}^{p} \alpha_i x_i, \quad \sum_{i=0}^{p} \alpha_i = 1, \quad \alpha_i \geq 0, \forall i, \quad (1)$$

where the $x_i$’s are the characteristic functions of downsets $X_i$ in $\mathcal{J}(L)$. Then $f$ has value 1 on $X_0$, value $1 - \alpha_0$ on $X_1 \setminus X_0$, value $1 - \alpha_0 - \alpha_1$ on $X_2 \setminus X_1$, etc., and value $\alpha_p$ on $X_p \setminus X_{p-1}$.

Example (ctd): Let us take $L = 2^N$, and consider a maximal chain in $\mathcal{D}(\mathcal{J}(L))$, denoted $C := \{1_{A_0} < 1_{A_1} < \cdots < 1_{A_n}\}$, where $1_{A_i}$ is the characteristic function of set $A_i \subseteq N$, and $\emptyset := A_0 \subset A_1 \subset \cdots \subset A_n := N$. For each such maximal chain (thus defining a $n$-dimensional simplex), there exists a permutation $\pi$ on $N$ such that $A_i = \{\pi(1), \ldots, \pi(i)\}$. Since $1_\emptyset \equiv 0$, we have for any $f \in \sigma(C)$:

$$f(j) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(j) = \sum_{A_i \supset j} \alpha_i, \quad \forall j \in N.$$ 

Observe that $f(\pi(1)) = 1 - \alpha_0, f(\pi(n)) = 0$, and in general $f(\pi(i)) = 1 - \sum_{j=0}^{i-1} \alpha_j$. Moreover, there are $n!$ $n$-dimensional simplices. □

3 Interpolative aggregation functions

Let us consider the index set $N := \{1, \ldots, n\}$, representing the indices of sources to be aggregated, and an aggregation function $F : E \to \mathbb{R}$. Usually, $E$ is taken as $[0, 1]^n$ or another closed real interval to the power $n$. We address here a more general construction.

For source $i$, let us denote by $L_i$ the set of remarkable levels pertaining to source $i$, assuming they are partially ordered by some order relation $\leq_i$, such that $(L_i, \leq_i)$ is a distributive lattice.

Let us give some examples. The simplest example is $L_i := \{0, 1\}$, i.e., remarkable levels are numerical values, representing the lowest and highest
outputs of source $i$. This lattice is usually denoted by $2$. Observe that in fact 0, 1 have no special numerical meaning since only order matters; hence we could have written $L_i := \{\bot, \top\}$ as well.

The next example is to take a finite chain $L_i := \{\bot, x_1, \ldots, x_k, \top\}$ with $\bot < x_1 < \cdots < x_k < \top$ (linear lattice). Levels $x_j$ may represent qualitative labels, such as \{bad, average, good\} or \{low, medium, high\}, etc.

We can also consider the following lattice:

```
  high
     medium
      don't know
  low
```

which is not a linear one. More complicated examples can be thought of, if sources provide several types of information at the same time.

The product lattice $L := L_1 \times \cdots \times L_n$ represents the possible combinations of all remarkable levels for all sources. Assuming that sources can output intermediate values between remarkable levels, the geometric realization of $L$ gives all possible outputs the sources can produce, and is therefore the domain $E$ of the aggregation function $F$.

The basic idea underlying interpolative aggregation functions is that the output of the aggregation is known (or fixed) for each element of $L$, and the output for any point of the geometric realization of $L$ is obtained as a linear interpolation between vertices defined by the natural triangulation. We summarize this in the following definition.

**Definition 1** Let $L := L_1 \times \cdots \times L_n$, with each $L_i$ being a distributive lattice representing the remarkable levels for source $i$. We consider an aggregation function $F : E \to \mathbb{R}$, where $E = \mathcal{C}(\mathcal{J}(L))$ is the geometric realization of $L$. $F$ is an interpolative aggregation function if $F|L$ is known (more precisely $F|_{\mathcal{D}(\mathcal{J}(L))}$), and for each $f \in E$,

$$F(f) = \sum_{i=0}^{p} \alpha_i x_i,$$

assuming that $f$ belongs to the interior of the simplex $\sigma(C)$, where $C := \{x_0 < x_1 < \cdots < x_p\}$ is a chain of $\mathcal{D}(\mathcal{J}(L))$, with $f = \sum_{i=0}^{p} \alpha_i x_i$.

**Example (ctd):** Consider again $L = 2^N$, and take the notations introduced before in this example. Expressing the $\alpha_i$’s in term of $f$, we get easily $\alpha_0 = 1 - f(\pi(1)), \ldots, \alpha_i = f(\pi(i)) - f(\pi(i + 1)), \ldots, \alpha_n = f(\pi(n))$. Then

$$F(f) = \sum_{i=1}^{n} \alpha_i F(1_{A_i})$$

$$= \sum_{i=1}^{n} [f(\pi(i)) - f(\pi(i + 1))] F(\{\pi(1), \ldots, \pi(i)\}),$$

with the convention $f(\pi(n + 1)) := 0$. Putting $\mu(A) := F(1_A)$, we recognize the Choquet integral $\int f \mu$. □

**Remark:** Usually, function $f$, which represents the output of sources, is given under the form $\hat{f} \in \prod_{i=1}^{n} \mathcal{C}(\mathcal{J}(L_i))$, i.e., the vector of output sources. In the above example, since $\mathcal{J}(L) = N$, we have $f \equiv \hat{f}$.

### 4 Bipolar structures

Let us start with a simple situation. We consider one source with three remarkable levels, say 0, 1, 2, representing the lowest, middle and highest outputs. A distinction can be introduced, depending on the precise meaning attached to the central level 1. A first situation is when the central level has no other meaning than simply a middle point on the scale, which we could call “medium”. This is the case if the source measures heights, lengths, masses, etc. A second situation is when some affect is attached to the levels. The central level has then a meaning of a neutral level or indifference level, and every output greater than this neutral level is felt as good or satisfactory, and every output smaller is felt as bad or unsatisfactory. This is the case if the source is a decision maker expressing some opinion.

In the first situation, we say that the scale of measurement is unipolar. There is no symmetry in this scale since the position of level 1 is somewhat arbitrary, while level 0 has a strong and absolute meaning since it represents the lowest possible output. Therefore, it is natural to assign the real value 0 to this level, i.e., $F(0) = 0$. 

In the second situation, the scale is said to be bipolar. Now level 1 is the center of symmetry of the scale, and to enhance this and distinguish from the unipolar case, we suggest changing the notation and denoting the levels by $-1$, 0 and 1. Hence, negative outputs pertain to bad outputs, and positive ones, to good outputs. We naturally set $F(0) = 0$. Observe that this can be seen as a symmetrization of the unipolar structure $\{0,1\}$ with two remarkable levels.

Let us consider now $n$ sources, being unipolar with remarkable levels 0, 1 and 2. The lattice $L := \{0,1,2\}^n$ endowed with the product order represents the set of all possible combinations of remarkable levels for all sources. Figure 1 illustrates the case $n = 2$. We represent each element $x$ by its coordinates $(x_1,x_2)$. Consider now that

![Figure 1: The lattice $\{0,1,2\}^2$](image)

all sources are bipolar, with levels $-1$, 0 and 1. Hence any combination of levels is an element of $\{-1,0,1\}^n$. Considering that $-1 \prec 0 \prec 1$ and taking the product order will lead to exactly the same structure as the unipolar one, just replacing 0,1,2 by $-1,0,1$. We consider instead that $\{-1,0,1\}^n$ is a symmetrization of $\{0,1\}^n$, in the following sense. Let us define

$$\widetilde{\{0,1\}} := \{(x,y) \in \{0,1\}^n \times \{0,1\}^n \mid x \land y = 0\}.$$ 

There is a bijection $\phi$ from $\{-1,0,1\}^n$ to $\{0,1\}^n \times \{0,1\}^n$ defined by:

$$x \mapsto \phi(x) := (x_1,x_2), \quad x_1 := x \lor 0, \quad x_2 := (x) \lor 0.$$ 

The left argument pertains to positive levels, while the right one pertains to negative levels. For example, taking $n = 5$, $x = (-1,1,1,0,-1)$ gives $\phi(x) = ((0,1,1,0,0),(1,0,0,0,1))$. The condition $x \land y = 0$ ensures that there is only one output per source: clearly, $((0,1,1),(0,0))$ corresponds to no element in $\{-1,0,1\}^5$. We endow $\{0,1\}^n$ with the product order of $\{0,1\}^n \times \{0,1\}^n$. Figure 2 illustrates the case $n = 2$. One can see that element $((0,0),(0,0))$ is the bottom of the structure, which is an inf-semilattice. It is much clearer to redraw this figure exactly as Fig. 1, and indicating by arrows the order relation (see Fig. 3). This shows clearly the construction of the bipolar structure: the original unipolar structure (top, in white) is duplicated and reversed (bottom, in grey), then combinations of positive and negative elements complete the structure (left and right, crossed). Observe the symmetry of arrows w.r.t. the horizontal line passing through the central point.

We are ready to present a general definition.

**Definition 2** Let us consider $(L,\leq)$ an inf-semilattice with bottom element $\bot$. The bipolar extension $\tilde{L}$ of $L$ is defined as follows:

$$\tilde{L} := \{(x,y) \mid x,y \in L, x \land y = \bot\},$$

which we endow with the product order $\leq$ on $L^2$.

Remark that $\tilde{L}$ is a downset (order ideal) of $L^2$. 

![Figure 2: The bipolar structure $\{0,1\}^n$ for $n = 2$](image)

![Figure 3: The bipolar extension](image)
The following holds.

**Proposition 1** Let \((L, \leq)\) be an inf-semilattice.

(i) \((\tilde{L}, \leq)\) is an inf-semilattice whose bottom element is \((\bot, \bot)\), where \(\leq\) is the product order on \(L^2\).

(ii) The set of join-irreducible elements of \(\tilde{L}\) is

\[\mathcal{J}(\tilde{L}) = \{(j, \bot) \mid j \in \mathcal{J}(L)\} \cup \{\langle \bot, j \rangle \mid j \in \mathcal{J}(L)\} \]

(iii) If \(L\) is a distributive lattice, the unique irredundant decomposition writes

\[(x, y) = \bigvee_{j \leq x, j \in \mathcal{J}(L)} (j, \bot) \lor \bigvee_{j \leq y, j \in \mathcal{J}(L)} (\bot, j).\]

We consider now the Möbius function over \(\tilde{L}\). The aim is to solve

\[f(x, y) = \sum_{(x', y') \leq (x, y), (x', y') \in \tilde{L}} g(x', y'), \ \forall (x, y) \in \tilde{L}, \]

where \(f, g\) are real-valued functions on \(\tilde{L}\). The solution is given through the Möbius function on \(\tilde{L}\):

\[g(x, y) = \sum_{(z, t) \leq (x, y), (z, t) \in \tilde{L}} f(z, t)\mu_{\tilde{L}}((z, t), (x, y)).\]

The following holds.

**Proposition 2** The Möbius function on \(\tilde{L}\) is given by:

\[\mu_{\tilde{L}}((z, t), (x, y)) = \mu_L(z, x)\mu_L(t, y).\]

Note that as usual, the set of functions \(u_{(x,y)}\) defined by

\[u_{(x,y)}(z, t) = \begin{cases} 1, & \text{if } (z, t) \geq (x, y) \\ 0, & \text{otherwise} \end{cases}\]

forms a basis of the functions on \(\tilde{L}\).

**Theorem 1** Let \(L\) be a finite distributive lattice, and \(c(L)\) be the set of its complemented elements. Then, for any \(x \in c(L)\), its complement being denoted by \(x'\), the subset \(L(x)\) of \(\tilde{L}\) defined by

\[L(x) := [(\bot, \bot), (x, x')]\]

and endowed with the product order of \(L^2\) is isomorphic to \(L\), by the order isomorphism \(\phi_x : L(x) \to L\), \((y, z) \mapsto y \lor z\). The inverse function \(\phi_x^{-1}\) is given by \(\phi_x^{-1}(w) = (w \land x, w \land x')\).

Moreover, the join-irreducible elements of \(L(x)\) are the image of those of \(L\) by \(\phi_x^{-1}\), i.e.

\[\mathcal{J}(L(x)) = \{(j \land x, j \land x') \mid j \in \mathcal{J}(L)\} \]

Remark that in any finite lattice, \(\bot\) and \(\top\) are complemented elements, and \(L(\top) = L\), and \(L(\bot) = L^*\), where \(L^*\) is the dual of \(L\). An interesting question is whether the union of all \(L(x)\), \(x \in c(L)\), is equal to \(\tilde{L}\).

**Theorem 2** Let \(L\) be a finite distributive lattice. Then the bipolar extension \(\tilde{L}\) can be written as:

\[\tilde{L} = \bigcup_{x \in c(L)} L(x)\]

if and only if \(L\) is a product of linear lattices.

This important result shows that \(\tilde{L}\) is composed by “tiles”, all identical to \(L\). Hence, an aggregation function on \(\tilde{L}\) can be constructed from any aggregation function \(F\) on \(L\), from a simple symmetrization. Let us denote by \(\tilde{F}\) the bipolar extension of \(F\) on \(L\). We detail the symmetrization procedure.

From now on, \(L\) is supposed to be a product of \(n\) linear lattices. Then

\[c(L) = \{(\top_A, \bot_{A'}) \mid A \subseteq N\} \]

where \((\top_A, \bot_{A'})\) has coordinate number \(i\) equal to \(\top_i\) if \(i \in A\), and \(\bot_i\) otherwise. Also, \((\top_A, \bot_{A'})' = (\bot_A, \top_{A'})\).

We consider some \(f\) in \(C(\mathcal{J}(L))\), such that \(f = \sum_{i=0}^p a_i x_i\), with \(x_0, \ldots, x_p\) forming a chain in \(D(\mathcal{J}(L))\). Given \(x \in c(L)\), let us define the corresponding \(f_x\) in \(L(x)\).

Recall that for any element \((y, z) \in L(x)\), \(y\) represents the positive part, and \(z\) the negative part of a particular combination of levels in our bipolar model. To each \((y, z) \in L(x)\) we assign a function \(\xi_{(y,z)} : \mathcal{J}(L(x)) \to \{-1, 0, 1\}\) by:

\[\xi_{(y,z)}(j \land x, j \land x') := \begin{cases} 1, & \text{if } j \in \eta(y) \\ -1, & \text{if } j \in \eta(z) \\ 0, & \text{otherwise} \end{cases}\]
for any \( j \in \mathcal{J}(L) \). Observe that \(|\xi|\) is nonincreasing. Reciprocally, to each function \( \xi : \mathcal{J}(L) \to \{-1, 0, 1\} \) such that \(|\xi|\) is nonincreasing, we can assign \((y_\xi, z_\xi)\) in \(L(x)\) by:

\[
y_\xi = \bigvee_{j \in \mathcal{J}(L(x))} \{ j \mid j \leq x \wedge j \leq x' \}, \quad z_\xi = \bigvee_{j \in \mathcal{J}(L(x))} \{ j \mid j \geq x \wedge j \geq x' \}.
\]

Calling \( \mathcal{D}(\mathcal{J}(L(x))) \) the set of such functions, we have established an isomorphism \( \psi \) between this set of functions and \(L(x)\):

\[
\psi : \mathcal{D}(\mathcal{J}(L(x))) \to L(x).
\]

We define \( f_x : \mathcal{J}(L)(x) \to [-1, 1] \) by

\[
f_x := \sum_{i=0}^{\frac{p}{\|x\|}} \alpha_i \psi^{-1}(\phi_x^{-1}(x_i))
\]

(with some abuse of notation, considering \( x_i \) as element of \( L \), up to an isomorphism). Explicitly, this gives, for any \( j \in \mathcal{J}(L) \):

\[
f_x(j \wedge x, j \wedge x') = \sum_{i \in \eta(x \wedge x)} \alpha_i - \sum_{i \in \eta(x \wedge x')} \alpha_i.
\]

Reciprocally, assume we know some \( \hat{f} \), output vector of sources. Since we are in the bipolar case, \( \hat{f} \) may take negative values. Let \( N_+ := \{ i \in N \mid \hat{f}_i \geq 0 \} \). Considering \(|\hat{f}|\), we are back to the unipolar case, and the corresponding \( f \) is an element of \(C(\mathcal{J}(L))\), which can be written as \( f = \sum_{i=0}^{p} \alpha_i x_i \), with \( x_0, \ldots, x_p \) a chain in \( \mathcal{D}(\mathcal{J}(L)) \). Defining \( x := (\bigvee_{N_+}, \bigwedge_{(N_+)^c}) \) (see (4)), we have that \( f_x \) defined above corresponds to \( \hat{f} \). Finally, we put, considering \( x_i \) as an element of \( L \):

\[
\tilde{F}(f_x) = \sum_{i=0}^{p} \alpha_i F_x(\psi^{-1}(x_i \wedge x, x_i \wedge x'))
\]

where \( F_x : \mathcal{D}(\mathcal{J}(L(x))) \to \mathbb{R} \) represents the known value of aggregation for elements of \(L(x)\).

Example (ctd): Let us take \( L = 2^N \) again. Since \( L \) is Boolean, any element \( B \in L \) is complemented. Then \( L = \{(A, B) \mid A, B \subseteq N, A \cap B = \emptyset\} \) (usually denoted by \( Q(N) \)), and \( L(B) = \{(A \cap B, A \cap B^c) \mid A \subseteq N\} \).

We already know that any \( f \in C(\mathcal{J}(L)) \) writes \( f(j) = \sum_{A_i \ni j} \alpha_i \), for some chain \( A_1 \supseteq \cdots \supseteq A_n \) associated to some permutation \( \pi \). We obtain

\[
f_B(j \cap B_j \cap B') = \sum_{A_i \ni B_j \ni B'} \alpha_i - \sum_{A_i \ni B_j \ni B'} \alpha_i.
\]

Now for a given output vector \( \hat{f} \) with \( N_+ \) defined as above, we have \( x = N_+ \), and we get:

\[
\tilde{F}(f_{N_+}) = \sum_{i=1}^{n} \alpha_i F_{N_+}(1_{A_i \cap N_+}, -1_{A_i \cap (N_+)^c})
\]

\[
= \sum_{i=1}^{n} \tilde{F}(\hat{f}_{(i)}) - \tilde{F}(\hat{f}_{(i+1)})
\]

Putting \( v(A, B) := F(1_A, -1_B) \), we recognize the Choquet integral for bi-capacities [2].

References