ALTERNATIVE REPRESENTATIONS OF
DISCRETE FUZZY MEASURES FOR DECISION MAKING

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This paper introduces three different representations of fuzzy measures, through the Möbius transformation, and the expression of importance and interaction. This leads naturally to the concept of $k$-order additive measures. It is shown how these concepts can be used in decision making, especially multicriteria evaluation.

Keywords: fuzzy measure; $k$-order additive measure; pseudo-Boolean function; Möbius inversion formula; Shapley value; interaction index; Choquet integral; multicriteria decision making

1. Introduction

In the last decades, several concepts of non-additive set functions have arisen independently in many different areas. Choquet is one of the first to be cited with his theory of capacities$^4$ proposed in 1953. Quite at the same time, Shapley introduced the same kind of set functions in the field of cooperative game theory.$^{27,1}$ In the seventies, Sugeno proposed the concept of fuzzy measure,$^{29}$ very similar to the one of capacity, and Shafer the concept of belief function,$^{26}$ based on previous works of Dempster.$^5$ Approximately at the same time, Zadeh proposed the idea of possibility measure,$^{33}$ later developed by Dubois and Prade$^9$ to become possibility theory. More recently, the decision making community has recognized the importance of what they call “non-additive probabilities”, through the pioneering works of Schmeidler,$^{24,25}$ Gilboa,$^{10}$ Wakker$^{31}$ and many others, who have introduced the non-additive expected utility model for decision making under uncertainty.

This effervescence around the concept of non-additive set function shows its importance and its necessity in the modelling of various phenomena related to artificial intelligence and decision in a broad sense. Coming back to fuzzy measures, Sugeno$^{29}$ introduced them to model, according to him, the subjective aspect of uncertainty, by contrast to the frequentist view, where probabilities hold an uncontested position since two centuries. It is important here to distinguish two separate understandings of fuzzy measures (and similar concepts), which will be denoted from now on by $\mu$. 

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• fuzzy measures can model uncertainty on the set $\Omega$ of states of the world, as probability usually does. Specifically, for any subset $A \subseteq \Omega$, $\mu(A)$ models the degree of belief (in the very rough and general sense of the term) that we have in the fact that the true state of the world belongs to $A$. Particular cases of fuzzy measures, such as possibility measures and belief functions have already been successfully applied in such contexts. This is typically what is done in decision making under uncertainty, e.g. in the non-additive expected utility model.

• fuzzy measure can model the importance of a coalition. Let us denote $X$ a set of elements, which could be players in a cooperative game, criteria in a multicriteria decision problem, attributes, experts or voters in an opinion pooling problem, etc. Then for any $A \subseteq X$, $\mu(A)$ represents the importance or strength of the coalition $A$ for the particular decision problem under consideration. This is typically what is done in cooperative game theory and multicriteria decision making.

This paper deals with both aspects, but putting emphasis on the modelling of importance of coalitions, due to the nature of the new concepts which will be introduced here.

Despite the fact that fuzzy measures, capacities and the like have been thoroughly studied on a mathematical point of view — see the monographs of e.g. Wang and Klu,32 Sugeno and Murofushi,30 Denneberg8, and Grabisch et al.17 —, one must admit that there is a considerable gap when looking at practical applications of general fuzzy measures (i.e. not restricted to possibility measures, belief functions, etc.), since what is found here is almost heuristic and poorly formalized. To our opinion, this essentially comes from two facts.

The first one is the exponential complexity involved by the manipulation of any set function, since they range over the power set. A further difficulty with fuzzy measures is that they must be monotonic with respect to set inclusion. This generates constraints which are difficult to handle when one tries to determine or optimize fuzzy measures. This is the reason why many people have used $\lambda$-measures or other kind of decomposable measures in practical applications. However, as it will be seen, these are far insufficient for the modelling of importance of coalitions.

The second fact is that it seems to be difficult to get an intuitive understanding of the very meaning of $\mu(A)$. Despite the fact that $\mu(A)$ is defined as the importance of coalition $A$, this does not help so much in practice, unless to restrict to universes with a very little number of elements. For example, we may have $\mu(\{i\}) = 0$, suggesting that element $i$ is unimportant, but it may happen that for many subsets $A \subseteq X$, $\mu(A \cup \{i\})$ is much greater than $\mu(A)$, suggesting that $i$ is actually important. The converse example may be built as well. This difficulty prevents from determining fuzzy measures only by elicitation from an expert.

This paper introduces new concepts in fuzzy measure theory which shed light on the interpretation of fuzzy measures, and help to solve the computational complexity
problem. The key concept is the one of representation of a fuzzy measure, that is, to transform fuzzy measures by linear irreversible operators. A typical example is the Möbius transform, known from a long time in combinatorics. 22 Another example introduced by the author is the interaction representation,14 which extends the Shapley value.27 This representation has a basic role in decision making.

The paper gathers and synthesizes results around this topic,12,14,15,16,18,23 and presents new ones. In a first part, (sections 2 to 5) basic notions are introduced together with general results. In a second part, (section 6) their application in decision making is presented.

In the whole paper, we will work on a finite universe \( X \) of \( n \) elements. \( P(X) \) is the power set of \( X \), while \( |A| \) denotes the cardinal of a set \( A \), \( A^c \) its complement, and \( A \setminus B \) denotes the set difference. \( \wedge, \vee \) denote min and max respectively.

2. Fuzzy measures and the Choquet integral

We give in this section only basic definitions restricted to the finite case. See the above cited monographs for more details.6,17,30,32

**Definition 1** (Sugeno,29 1974) A fuzzy measure \( \mu \) on \( X \) is a function \( \mu : P(X) \rightarrow [0,1] \), satisfying the following axioms.

(i) \( \mu(\emptyset) = 0, \mu(X) = 1 \).

(ii) \( A \subset B \subset X \) implies \( \mu(A) \leq \mu(B) \).

Usually, it is not necessary to enforce \( \mu(X) = 1 \), but this is done here due to the application field. Fuzzy measures are thus non negative monotonic set functions on \( X \). It is to be noted that the representations which will be introduced in the next section are applicable to any set function.

For any fuzzy measure \( \mu \), the dual fuzzy measure \( \mu^* \) is defined by \( \mu^*(A) = 1 - \mu(A^c) \) for any \( A \subset X \).

By referring to game theory, a fuzzy measure is said to be a unanimity game \( u_A \) for subset \( A \subset X \) when \( u_A(B) = 1 \) if and only if \( B \supset A \), and is zero otherwise. These are also called simple belief functions or 0-1 necessity measures. They play an important role since they form a basis of the set of fuzzy measures (see section 3.1).

**Definition 2** (Choquet,4 1953) Let \( \mu \) be a fuzzy measure on \( X \), whose elements are denoted \( x_1, \ldots, x_n \) here. The discrete Choquet integral of a function \( f : X \rightarrow \mathbb{R}^+ \) with respect to \( \mu \) is defined by

\[
(C) \quad \int f \, d\mu := \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(A_{(i)}),
\]

where \( x_{(i)} \) indicates that the indices have been permuted so that \( 0 \leq f(x_{(1)}) \leq \cdots \leq f(x_{(n)}) \), and \( A_{(i)} := \{x_{(i)}, \ldots, x_{(n)}\} \), and \( f(x_{(0)}) = 0 \).
The Choquet integral will be sometimes denoted $C_\mu$, in an operator-like way. The
definition can be extended to real functions by the formula

$$(C) \int f \, d\mu = (C) \int f^+ \, d\mu - (C) \int f^- \, d\mu,$$

where $f^+$, $f^-$ are the positive and negative parts of $f = f^+ - f^-$.  

3. Three representations of a fuzzy measure

3.1. Pseudo-Boolean functions and Möbius transform

In the field of complexity analysis, pseudo-Boolean functions are often used. A
pseudo-Boolean function is simply a real valued function $f : \{0,1\}^n \rightarrow \mathbb{R}$ (see e.g.
Hammer and Holzmann). It is easy to see that a fuzzy measure is a particular case
of pseudo-Boolean function: simply remark that for any $A \subset X$, $A$ is equivalent to
a point $(x_1, \ldots, x_n)$ in $\{0,1\}^n$ such that $x_i = 1$ iff $i \in A$. It can be shown that any
pseudo-Boolean function can be put under a multilinear polynomial in $n$ variables:

$$f(x) = \sum_{T \subset X} a_T \prod_{i \in T} x_i,$$

with $a_T \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \{0,1\}^n$. As $\prod_{i \in T} x_i$ corresponds to unanimity
game $u_T$, this result shows clearly that unanimity games form a basis of the set of
fuzzy measures.

The coefficients $a_T$’s, which are the coordinates of $f$ in the basis of unanimity
games, have another interpretation in the field of combinatorics.

**Definition 3** Let $\mu$ be a set function (not necessarily a fuzzy measure) on $X$. The
Möbius transform of $\mu$ is a set function on $X$ defined by

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad \forall A \subset X.$$  

The transformation is invertible, and $\mu$ can be recovered from $m$ by

$$\mu(A) = \sum_{B \subset A} m(B), \quad \forall A \subset X.$$  

It is clear that the coefficient $a_T$ corresponds to $m(T)$, $\forall T \subset X$.

3.2. Shapley value and interaction index

Relying on the interpretation of a fuzzy measure in terms of importance of
coalitions, we could say that the importance of an element $i$ is simply expressed
by the value of $\mu(\{i\})$ alone. In fact, as the example in the introduction suggests,
all values $\mu(A)$ such that $i \in A$ must also be taken into account. Shapley has
proposed a definition of a coefficient of importance, based on a set of reasonable
axioms.
Definition 4 Let \( \mu \) be a fuzzy measure on \( X \). The Shapley index for every \( i \in X \) is defined by
\[
v_i := \sum_{K \subseteq X \setminus \{i\}} \gamma(K) [\mu(K \cup \{i\}) - \mu(K)],
\]
with
\[
\gamma_k := \frac{(n - k - 1)!k!}{n!}.
\]
The Shapley value of \( \mu \) is the vector \( v(\mu) = [v_1 \cdots v_n] \).

The Shapley index \( v_i \) can be interpreted as a kind of average value of the contribution of element \( i \) alone in all coalitions. Basic properties of the Shapley value are
\[
\sum_{i=1}^{n} v_i = \mu(X) \quad \text{and} \quad v^{\mu + \nu} = v^\mu + v^\nu,
\]
where \( v^\nu \) indicates the Shapley value of \( \nu \).

Another interesting concept is the one of interaction between elements. Taking two elements \( i, j \), the value of \( \mu(\{i, j\}) \) could be the sum of \( \mu(\{i\}) \) and \( \mu(\{j\}) \). In this case, the individual importances are adding without interfering, and there is no interaction between \( i \) and \( j \). If \( \mu(\{i, j\}) \) is lower (resp. greater) than \( \mu(\{i\}) + \mu(\{j\}) \), then \( i \) and \( j \) interfere in a negative (resp. positive) way. As for importance, a proper definition should consider not only \( \mu(\{i\}), \mu(\{j\}), \mu(\{i, j\}) \) but also the measures of all subsets containing \( i \) and \( j \). Murofushi and Soneda\(^{21}\) have proposed the following definition, borrowing concepts from multiattribute utility theory,\(^{20}\) which is very similar to the one of Shapley.

Definition 5 Let \( \mu \) be a fuzzy measure on \( X \). The interaction index of elements \( i, j \) is defined by
\[
I_{ij} := \sum_{K \subseteq X \setminus \{i, j\}} \zeta(K) [\mu(K \cup \{i, j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)],
\]
with
\[
\zeta_k := \frac{(n - k - 2)!k!}{(n - 1)!}.
\]
The interaction index \( I_{ij} \) can be interpreted as a kind of average value of the added value given by putting \( i \) and \( j \) together, all coalitions being considered. When \( I_{ij} \) is positive (resp. negative), then the interaction is said to be positive (resp. negative).

3.3. Three representations of a fuzzy measure

Section 3.1 suggests that fuzzy measures can be given unambiguously by their Möbius transform, since the inverse transform exists. One may remark also that \( v_i \) can be viewed as the value taken by a set function on singleton \( i \), while \( I_{ij} \) would be the value of a set function on the pair \( i, j \). Grabisch\(^{14}\) has extended this idea by proposing an interaction index for any subset \( A \) of elements.
\[
I(A) := \sum_{B \subseteq X \setminus A} \xi(B, A) \sum_{C \subseteq A} (-1)^{|A \setminus C|} \mu(C \cup B),
\]
with
\[
\xi(B, A) := \frac{(n - |B| - |A|)!|B|!}{(n - |A| + 1)!}.
\]
It is clear that this is a generalization of both the Shapley value and the interaction index of Murofushi and Soneda, since \( v_i \) coincides with \( I(\{i\}) \) and \( I_{ij} \) with \( I(\{i,j\}) \). For this reason, \( I(A) \) can be said to be the interaction index for elements among \( A \). Moreover, \( I \) is also a set function defined for any \( A \subset X \), which happens to be irreversible (see below), so that \( I \) can be viewed as a kind of transform, similarly to the Möbius transform.

We may call representation of a fuzzy measure \( \mu \) (or more generally, of a set function), any set function \( \nu \) from which it is possible to recover \( \mu \) without loss of information.

In summary, we have found the following representations of a fuzzy measure:

- usual (measure) representation (this is simply \( \mu \) itself)
- Möbius representation \( m \) or polynomial representation \( a \)
- interaction representation \( I \).

Formulas between the different representations have been established.\(^\text{16,7} \). They are listed below.

\[
I(A) = \sum_{B \subset X \setminus A} \frac{1}{|B| + 1} m(A \cup B),
\]

(8)

\[
m(A) = \sum_{B \subset X \setminus A} \alpha_i I(B \cup A),
\]

(9)

\[
\mu(A) = \sum_{B \subset X} \beta_{|A \cap B|} I(B),
\]

(10)

with

\[
\alpha_k := -\sum_{l=0}^{k-1} \frac{\alpha_l}{k-l+1} \binom{k}{l}.
\]

(11)

and \( \alpha_0 = 1 \). In fact, the coefficients \( \alpha_k \) are the Bernoulli numbers, usually denoted \( B_k \). First terms of the sequence are \( \alpha_1 = -1/2, \alpha_2 = 1/6, \alpha_3 = 0, \alpha_4 = -1/30, \alpha_5 = 0 \).

\[
\beta_{k}^{l} := \sum_{j=0}^{k} \binom{k}{j} \alpha_{l-j}.
\]

(12)

First values of \( \beta_{k}^{l} \) are:

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tr>
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<td>0</td>
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</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-\frac{1}{30}</td>
</tr>
</tbody>
</table>
The Bernoulli numbers and the coefficients \( \beta_k \) possess remarkable properties (see appendix).

### 3.4. Other representations

Obviously, the above three representations of a fuzzy measure are not the only conceivable ones. Actually, any (preferably linear) irreversible transformation can do, at the condition that there is some interpretation behind (e.g. in decision making). Roubens\textsuperscript{23,18} has proposed some new ones, taking as starting point the Banzhaf index\textsuperscript{2} in game theory, which is very similar to the Shapley value. It is defined by

\[
b_i := \frac{1}{2^{n-1}} \sum_{A \subseteq X \setminus \{i\}} [\mu(A \cup \{i\}) - \mu(A)],
\]

and leads by generalization to the following notion of interaction:

\[
J(A) := \frac{1}{2^{n-|A|}} \sum_{B \subseteq X \setminus A} \sum_{C \subseteq A} (-1)^{|A \setminus C|} \mu(C \cup B).
\]

Compared to \( I \), \( J \) is a non-weighted average of differences of \( \mu \). It does not take into account the number of elements in subsets. Also, contrarily to \( v_i \), \( \sum_{i=1}^n b_i \neq \mu(X) \) in general.

A last representation should be mentionned, which has its origin in Shafer’s theory of evidence.\textsuperscript{26} We call it here the co-measure representation, denoted \( \bar{\mu} \).

**Definition 6** The co-measure representation of \( \mu \), denoted \( \bar{\mu} \), is defined by:

\[
\bar{\mu}(A) := \sum_{B \supset A} m(B), \forall A \subset X.
\]

Note that \( \bar{\mu}(\emptyset) = 1 \), and \( \mu(A) + \bar{\mu}(A) - m(A) = 1 \), for every \( A \subset X \). The comparison of (4) and (15) justifies the term “co-measure”. Note that a monotonic measure will have a anti-monotonic co-measure, i.e. \( \mu(A) \geq \bar{\mu}(B) \) whenever \( A \subset B \). When \( \mu \) is a belief function, \( \bar{\mu} \) is called the commonality function by Shafer.

Let us express \( \bar{\mu} \) in terms of \( I \), since this will be useful in the sequel.

**Property 1** Let \( \mu \) be a set function, \( \bar{\mu}, I \) its co-measure and interaction representations. Then

\[
\bar{\mu}(A) = \sum_{B \subset X \setminus A} \beta_{|B|} I(A \cup B).
\]

**Proof.** Using (9), we get

\[
\bar{\mu}(A) = \sum_{B \subset X \setminus A} m(A \cup B)
\]

\[
= \sum_{B \subset X \setminus A} \sum_{C \subset X \setminus (A \cup B)} \alpha_{|C|} I(A \cup B \cup C).
\]
Rearranging terms, we get
\[ \mu(A) = \sum_{B \subseteq X \setminus A} I(A \cup B) \left[ \sum_{l=0}^{[B]} \binom{[B]}{l} \alpha_{[B]-l} \right]. \]

The term into brackets being \( \beta_{[B]}^{[B]} \), this completes the proof. □.

It is proven in the appendix (property 2 (ii)) that \( \beta_k = \alpha_k \), for all \( k \geq 0, k \neq 1 \), and \( \beta_1 = -\alpha_1 = \frac{1}{2} \).

4. \( k \)-order additive measures

This concept was proposed by Grabisch, in an attempt to solve the problem of exponential complexity of fuzzy measures. Looking at the polynomial expression of a fuzzy measure (2), one can notice that additive measures have a linear representation \( f(x) = \sum_{i=1}^{n} a_i x_i \). By extension, we may think of a fuzzy measure having a polynomial representation of degree 2, or 3, or any fixed integer \( k \). It is natural to call these fuzzy measures \( k \)-order additive fuzzy measures or simply \( k \)-additive measures. But this means that the Möbius transform of such measures is 0 for subsets of more than \( k \) elements. In summary, we propose the following definition.

**Definition 7** A fuzzy measure \( \mu \) is said to be \( k \)-order additive or simply \( k \)-additive if its Möbius transform \( m(A) = 0 \) for any \( A \) such that \( |A| > k \), and there exists at least one subset \( A \) of \( X \) of exactly \( k \) elements such that \( m(A) \neq 0 \).

The following property of \( k \)-additive measures is easy to prove from formulas (8), (15), and (14).

**Property 2** Let \( \mu \) be a \( k \)-additive measure on \( X \). Then

(i) \( I(A) = \mu(A) = 0 \) for every \( A \subseteq X \) such that \( |A| > k \),

(ii) \( I(A) = J(A) = m(A) = \mu(A) \) for every \( A \subseteq X \) such that \( |A| = k \),

(iii) if \( \mu \) is a 2-additive measure, \( I(A) = J(A) \) for every \( A \subseteq X \).

Thus, \( k \)-additive measures can be represented by a limited set of coefficients, either \( m(A), |A| \leq k \), or \( I(A), |A| \leq k \), or equivalently \( \mu(A), J(A), |A| \leq k \) i.e. at most \( \sum_{i=1}^{k} \binom{n}{i} \) coefficients.

5. Properties of the representations

We give here essential properties of the different representations, limited to what is useful in the sequel. More properties can be found in the above cited references, together with the proofs, \( 7,16,18 \)

**Property 3** Let \( \mu, \mu^* \) be a pair of dual measures, and consider their Möbius representations \( m, m^* \) respectively. Then for any \( A \subseteq X \),

\[ m^*(A) = (-1)^{|A|+1} \sum_{B \supset A} m(B). \]
Property 4 Let \( \mu, \mu^* \) be a pair of dual measures, and consider their interaction representations \( I, I^* \) respectively. Then for any \( \emptyset \neq A \subseteq X \),

\[
I^*(A) = (-1)^{|A|+1} I(A).
\]

\[
J^*(A) = (-1)^{|A|+1} J(A).
\]

Property 5 Let \( \mu \) be a fuzzy measure. Then the interaction index \( I_{ij} \) ranges in \([-1, +1]\). \( I_{ij} = 1 \) if and only if \( \mu = u_{(i,j)} \), the unanimity game for the pair \( i, j \). Similarly, \( I_{ij} = -1 \) if and only if \( \mu = u_{(i,j)}^* \), the dual measure of the unanimity game for the pair \( i, j \).

The three following properties concern what could be called an inverse problem. Given a set of real coefficients \( \{c_A\}_{A \subseteq X} \), we may consider them as the Möbius or the interaction representation of an (unknown) fuzzy measure \( \mu \). Of course, one can compute the corresponding \( \mu \) using (4) and (10), but nothing ensures that \( \mu \) will be indeed a fuzzy measure, that is, monotonic and satisfying \( \mu(\emptyset) = 0, \mu(X) = 1 \). Constraints which are necessary and sufficient for obtaining a proper fuzzy measure have been investigated. The first result is due to Chateauneuf and Jaffray,\(^3\) the second one to Grabisch,\(^1,6\) and the third one to Roubens.\(^18\)

Property 6 A set of \( 2^n \) coefficients \( m(A), A \subseteq X \) corresponds to the Möbius representation of a fuzzy measure if and only if

(i) \( m(\emptyset) = 0, \sum_{A \subseteq X} m(A) = 1 \),

(ii) \( \sum_{i \in B \subseteq A} m(B) \geq 0 \), for all \( A \subseteq X \), for all \( i \in A \).

Property 7 A set of \( 2^n \) coefficients \( I(A), A \subseteq X \) corresponds to the (Shapley) interaction representation of a fuzzy measure if and only if

(i) \( \sum_{A \subseteq X} \alpha_{|A|} I(A) = 0 \),

(ii) \( \sum_{i \in X} I(\{i\}) = 1 \),

(iii) \( \sum_{A \subseteq X \setminus i} \beta_{|A|} |B| I(A \cup \{i\}) \geq 0, \forall i \in X, \forall B \subseteq X \setminus \{i\} \).

Property 8 A set of \( 2^n \) coefficients \( J(A), A \subseteq X \) corresponds to the (Banzhaf) interaction representation of a fuzzy measure if and only if

(i) \( \sum_{A \subseteq X} \left(-\frac{1}{2}\right)^{|A|} J(A) = 0 \),

(ii) \( \sum_{A \subseteq X} \left(\frac{1}{2}\right)^{|A|} J(A) = 1 \),

(iii) \( \sum_{A \subseteq X \setminus i} \left(\frac{1}{2}\right)^{|A|} (\pm 1)^{|A|} (\pm 2)^{|B|} J(A \cup \{i\}) \geq 0, \forall i \in X, \forall B \subseteq X \setminus \{i\} \).

The following properties express the Choquet integral in the various representations.

The result concerning \( m \) has been shown by Chateauneuf and Jaffray,\(^3\) — also one
can find particular cases in Dempster,\(^5\) the result concerning \(I\) and \(\mu\) has been shown by Roubens,\(^1^8\) while Grabisch\(^1^5\) showed the one concerning \(I\).

**Property 9** Let \(\mu\) be a fuzzy measure, \(m, \mu\) their Möbius and co-measure representations. Then the discrete Choquet integral of \((t_1, \ldots, t_n)\) w.r.t. \(\mu\) is expressed by:

\[
C_\mu(t_1, \ldots, t_n) = \sum_{A \subseteq X} m(A) \bigwedge_{i \in A} t_i, \tag{17}
\]

\[
C_\mu(t_1, \ldots, t_n) = \left( \sum_{A \subseteq X, A \neq \emptyset} (-1)^{|A|+1} \mu(A) \bigvee_{i \in A} t_i \right). \tag{18}
\]

Remark that the linear form (up to a reordering) of the Choquet integral in (1) has turned to a sum of maxima in the case of the co-measure representation, and to a sum of minima for the Möbius representation.

**Property 10** Let \(\mu\) be a fuzzy measure on \(X\), and \(I\) its (Shapley) interaction representation. Then the Choquet integral with respect to \(\mu\) can be written as

\[
C_\mu(t_1, \ldots, t_n) = \sum_{A \subseteq X} \left[ \sum_{B \subseteq X \setminus A} \alpha_{|B|} I^+(A \cup B) \right] \bigwedge_{i \in A} t_i + \sum_{A \subseteq X, A \neq \emptyset} (-1)^{|A|+1} \left[ \sum_{B \subseteq X \setminus A} \beta_{|B|} I^-(A \cup B) \right] \bigvee_{i \in A} t_i, \tag{19}
\]

with \(I^+\) indicating a restriction so that only terms with positive interaction are taken into account, and similarly for \(I^-\).

**Proof.** We define the following set functions \(\mu^+, \mu^-\) by their interaction representations denoted

\[
I^{\mu^+}(A) = \begin{cases} I(A), & \text{if } I(A) \geq 0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
I^{\mu^-}(A) = \begin{cases} I(A), & \text{if } I(A) \leq 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Due to the linearity of \(I\) on set functions, we have \(\mu = \mu^+ + \mu^-\). Now for any \((t_1, \ldots, t_n) \in \mathbb{R}^n\), we can write

\[
C_\mu(t_1, \ldots, t_n) = C_{\mu^+}(t_1, \ldots, t_n) + C_{\mu^-}(t_1, \ldots, t_n),
\]

since for a fixed ordering of the \(t_i\)’s, \(C_\mu\) is linear in \(\mu\). Using formulas (17) and (18), we get

\[
C_\mu(t_1, \ldots, t_n) = \sum_{A \subseteq X} m^{\mu^+}(A) \bigwedge_{i \in A} t_i + \sum_{A \subseteq X} (-1)^{|A|+1} \mu^- (A) \bigvee_{i \in A} t_i,
\]

with \(m^{\mu^+}\) and \(\mu^-\) the Möbius representation of \(\mu^+\) and the co-measure representation of \(\mu^-\). Using formulas (9) and (16), we get immediately the desired result. \(\square\).
Property 11 Let $\mu$ be non-additive measure on $X$. The Choquet integral with respect to $\mu$ in the $J$ representation is

\[
C_\mu(t_1, \ldots, t_n) = \sum_{A \subseteq X} \left[ \sum_{B \subseteq X \setminus A} \left( -\frac{1}{2} \right)^{|B|} J^+(A \cup B) \right] \bigwedge_{i \in A} t_i + \sum_{A \subseteq X} (-1)^{|A|+1} \left[ \sum_{B \subseteq X \setminus A} \left( \frac{1}{2} \right)^{|B|} J^-(A \cup B) \right] \bigvee_{i \in A} t_i,
\]

with $J^+$, $J^-$ defined similarly as in Property 10.

6. Application to decision making

As said above, fuzzy measures can be considered either as a general class of uncertainty measures, encompassing probabilities, or as a mean of modelling importance or strength of coalitions. In decision making, the first aspect concerns decision in uncertain environment, while the second one concerns multicriteria decision making, multiperson decision making, cooperative game theory, etc. We will illustrate here the two aspects, although putting emphasis on the second one.

6.1. Modelling of uncertainty

Let us consider first fuzzy measures as general uncertainty measures. One of the main questions in this field is the relation to probabilities. This occurs when fuzzy measures are lower or upper envelopes of a family of probabilities (ill-specified probability), or when one is looking for a suitable probability compatible with a fuzzy measure. This last case is frequent in evidence theory and possibility theory: a probability $P$ is said to be compatible with a pair of belief and plausibility functions $(\text{Bel}, \text{Pl})$ (or a pair of possibility and necessity measures $(\text{II}, \text{N})$), if for any $A \subseteq X$ we have $\text{Bel}(A) \leq P(A) \leq \text{Pl}(A)$ (similarly for $\text{II}$, $\text{N}$). Dempster\(^5\) has shown that for a given belief function Bel, with associated Möbius representation $m$, any compatible probability is of the form:

\[ P\{i\} = \sum_{B \ni i} \lambda(B, i), \]

with $\lambda(B, i) \in [0, 1]$ for any $B \subseteq X$, $i \in X$, and $m(B) = \sum_{i \in B} \lambda(B, i)$. In other words, for a given $B$, the coefficients $\lambda(B, i), i \in B$ represent one possible sharing out of the mass attributed to $B$ to all singletons inside $B$. In the absence of any further information, the most natural way to share out the total mass of $B$ is to divide it equally to all elements, that is $\lambda(B, i) = m(B)/|B|$ for every $i \in B$. Then we get easily

\[ P\{i\} = m\{i\} + \frac{1}{2} \sum_j m\{i, j\} + \frac{1}{3} \sum_{j,k} m\{i, j, k\} + \cdots + \frac{1}{n} m(X). \]

This is called by Smets the pignistic transformation\(^{28}\). Comparing with (8), we see that $P\{i\}$ is nothing else than the Shapley value $u_i$. 
6.2. Multicriteria decision making

Let us now turn to the modelling of strength or importance of coalitions by fuzzy measures in a multicriteria decision making problem. We consider here that elements in $X$ are criteria.

As explained in section 3.2, the Shapley value expresses the global importance of each criterion, taking into account the effect of each criterion into all coalitions. Moreover, it gives the sharing of the total importance of all criteria $\mu(X)$ among them. Note that although the Banzhaf index expresses also the global importance of each criterion, it does not represent a sharing of $\mu(X)$ since in general $\sum_{i=1}^{n} b_i \neq \mu(X)$. This means that in a multicriteria problem, the Shapley value seems to be the right concept for finding important criteria or discard unimportant ones.

Similarly, the interaction index $I_{ij}$ models the interaction of criteria $i, j$. It is positive when $i, j$ act in a cooperative or complementary way (i.e. the importance of the pair $(i, j)$ is significantly greater than importances of $i$ and $j$ alone), and it is negative when $i, j$ act in a redundant or substitutive way (i.e. the importance of the pair $(i, j)$ is almost the same as importances of $i$ and $j$ alone). The value of $I(A)$ when $|A| > 2$ can be interpreted in the same way, but it becomes difficult to grasp intuitively the very meaning of it. Thus, for the purpose of a semantical analysis, we may consider that it is sufficient to restrict to $v_i$ and $I_{ij}$, i.e. to restrict to 2-additive measures.

We can further explain the meaning of these interaction indices by means of the Choquet integral. In multicriteria decision making, the discrete Choquet integral is used as an aggregation operator to compute the overall evaluation of an alternative with respect to several criteria, the fuzzy measure modelling the importance of coalitions of criteria. Specifically, denoting $t_1, \ldots, t_n$ the scores or performances of a given alternative with respect to $n$ criteria, the global score $t$ of the alternative is computed by the discrete Choquet integral

$$t = C_\mu(t_1, \ldots, t_n),$$

where $\mu$ is a fuzzy measure on the set of criteria (see Grabisch\textsuperscript{11,13} for a study and a justification of this kind of aggregation in multicriteria decision making).

Let us consider 2-additive measures, and express formula (19) in this case. We obtain

$$C_\mu(t_1, \ldots, t_n) = \sum_{I_j > 0} (t_i \wedge t_j) I_{ij} + \sum_{I_j < 0} (t_i \vee t_j) |I_{ij}| + \sum_{i=1}^{n} t_i (v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}|),$$

(21)

with the property that $v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}| \geq 0$ for all $i$ (this inequality comes from Property 7 (iii): take $B := \{j \in X | I_{ij} < 0\}$). It can be seen that the Choquet integral for 2-additive measures can be decomposed in a conjunctive, a disjunctive and an additive part, corresponding respectively to positive interaction indices, negative interaction indices, and the Shapley value. This makes clear the precise meaning of $I_{ij}$ in the framework of the Choquet integral:
• a positive $I_{ij}$ implies a conjunctive behaviour between $i$ and $j$. This means that the simultaneous satisfaction of criteria $i$ and $j$ is significant for the global score, but a unilateral satisfaction has no effect.

• a negative $I_{ij}$ implies a disjunctive behaviour, which means that the satisfaction of either $i$ or $j$ is sufficient to have a significant effect on the global score.

• the Shapley value acts as a weight vector in a weighted arithmetic mean. This represents the linear part of Choquet integral. It will be small if interaction indices are large.

The case of 3-additive measures leads to a more complicated equation for the Choquet integral, which is less easy to interpret. We obtain

$$C_\mu(t_1, \ldots, t_n) = \sum_{i \in X} t_i [I_i - \frac{1}{2} \sum_{j \in X \setminus i} (I_{ij}^+ - I_{ij}^-) + \frac{1}{6} \sum_{\{i,j,k\} \subseteq X} (I_{ijk}^+ + I_{ijk}^-)]$$

$$+ \sum_{\{i,j\} \subseteq X} (t_i \wedge t_j) [I_{ij}^+ - \frac{1}{2} \sum_{k \in X \setminus \{i,j\}} I_{ijk}^+] + \sum_{\{i,j,k\} \subseteq X} (t_i \wedge t_j \wedge t_k) I_{ijk}^+$$

$$+ \sum_{\{i,j\} \subseteq X} (t_i \vee t_j) [-I_{ij}^- - \frac{1}{2} \sum_{k \in X \setminus \{i,j\}} I_{ijk}^-] + \sum_{\{i,j,k\} \subseteq X} (t_i \vee t_j \vee t_k) I_{ijk}^-.$$  

The interpretation, which seems to remain similar to the 2-additive case — i.e. positive (negative) interactions entail conjunction (disjunction) —, is obscured by the fact that coefficients in front of $t_i$’s are not always positive.

Another important topic in multicriteria decision making is the concept of veto and favor.

**Definition 8** Suppose $\mathcal{H}$ is an aggregation operator being used for a multicriteria decision making problem. A criterion $i$ is a veto for $\mathcal{H}$ if for any $n$-uple $(t_1, \ldots, t_n) \in \mathbb{R}^+ \land \mathbb{R}$ of scores,

$$\mathcal{H}(t_1, \ldots, t_n) = t_i \wedge \mathcal{G}(t_1, \ldots, t_n),$$

where $\mathcal{G}$ is another aggregation operator. Similarly, criterion $i$ is a favor for $\mathcal{H}$ if for any $n$-uple $(t_1, \ldots, t_n)$ of scores,

$$\mathcal{H}(t_1, \ldots, t_n) = t_i \vee \mathcal{G}(t_1, \ldots, t_n).$$

This means that when criterion $i$ is a veto, if the score on $i$ is high, it has no effect on the evaluation, but if it is low, the global score will be low too, whatever the values of the other scores are. The concepts of veto and favor have been already proposed by Dubois and Koning in the context of social choice functions, where “favor” was called “dictator”.

Let us take for $\mathcal{H}$ the Choquet integral and show that these veto and favor effects can be represented by a suitable fuzzy measure.
Property 12 For the Choquet integral, i is a veto if and only if the fuzzy measure satisfies \( \mu(A) = 0 \) whenever \( i \not\in A \). Such fuzzy measures are denoted \( \mu^\land \). Similarly, i is a favor if and only if the fuzzy measure satisfies \( \mu(A) = 1 \) whenever \( i \in A \). Such fuzzy measures are denoted \( \mu^\lor \). Moreover, \((\mu^\land)^*\) has a favor effect for i and \((\mu^\lor)^*\) a veto effect for i.

Proof, Let us show the result for the veto effect, the case of the favor effect proceeds similarly. Using the definition of \( C_\mu \), rearranging terms, defining \( A_{(n+1)} = \emptyset \), and denoting \( \mu(A) \) by \( \mu_A \), we have, supposing without loss of generality that \( t_i = t_{(k)} \),

\[
C_\mu^\land (t_1, \ldots, t_n) = \sum_{j=1}^{k-1} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)}) + t_{(k)} (\mu_{A(k)} - \mu_{A(k+1)}) + \sum_{j=k+1}^{n} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)})
\]

\[
\leq \sum_{j=1}^{k-1} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)}) + t_{(k)} (\mu_{A(k)} - \mu_{A(k+1)}) + \sum_{j=k+1}^{n} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)})
\]

\[
= t_i - t_i \mu_{A(k+1)} + \sum_{j=k+1}^{n} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)})
\]

i is a veto if and only if \( C_\mu(t_1, \ldots, t_n) \leq t_i \) for every n-uple \((t_1, \ldots, t_n)\) in \( \mathbb{R}^+ \). From the above, we deduce that i is a veto if and only if

\[-t_i \mu_{A(k+1)} + \sum_{j=k+1}^{n} t_{(j)} (\mu_{A(j)} - \mu_{A(j+1)}) = 0,
\]

for every n-uple \((t_1, \ldots, t_n)\) in \( \mathbb{R}^+ \). The only solution is then to have factors in front of the \( t_k \)'s equal to 0. This leads to

\[
\mu_{A(k+1)} = \mu_{A(k+2)} = \cdots = \mu_{A(n)} = 0,
\]

This must be true for any ordering of the sequence \( t_1, \ldots, t_n \), implying that \( \mu(A) = 0 \) whenever \( i \not\in A \). Conversely, if \( \mu(A) = 0 \) whenever \( i \not\in A \), the above equations are satisfied. \( \square \)

Remark that interpreting fuzzy measures as strength of coalitions, the definitions of \( \mu^\land \) and \( \mu^\lor \) seem to be natural. Also, remark that if criterion i is both a veto and a favor, then it is a dictator, i.e. \( C_\mu(t_1, \ldots, t_n) = t_i \). Another consequence of the definition is that for a given \( H \), it is not possible to have simultaneously a veto on i and a favor on j, \( j \neq i \) since having \( H(t_1, \ldots, t_n) \leq t_i \) and \( H(t_1, \ldots, t_n) \geq t_j \) is not compatible in general.

Let us examine the interaction representation of \( \mu^\land \) and \( \mu^\lor \). Denoting \( I^\land_{jk} \) and \( I^\lor_{jk} \) their respective interaction indices, it is easy to show that (use (5))

\[
I^\land_{jk} \geq 0, \quad I^\lor_{jk} \leq 0, \quad \forall k \neq i.
\]

Remark also that the same property holds for \( J \), the Banzhaf interaction index. Property 12 and equation (22) show that if i is a veto, then necessarily \( I_{ik} \geq 0 \),
for any $k \neq i$ (similarly for a favor), but this is not a sufficient condition. Simple results can be given for 2-additive measures. We can show the following.

**Property 13** Let $\mu$ be a 2-additive measure. Criterion $i$ is a veto for the Choquet integral if and only if the following conditions are satisfied

(i) $I_{ik} \geq 0, \ \forall k \neq i,$

(ii) $I_{kl} = 0, \ \forall k, l \neq i,$

(iii) $v_k = \frac{1}{2} I_{ik}, \ \forall k \neq i.$

Similarly, $i$ is a favor if and only if

(i) $I_{ik} \leq 0, \ \forall k \neq i,$

(ii) $I_{kl} = 0, \ \forall k, l \neq i,$

(iii) $v_k = -\frac{1}{2} I_{ik}, \ \forall k \neq i.$

**Proof**, Criterion $i$ is a veto if and only if $\mu$ is of the $\mu^{i\wedge}$ type. Under the hypothesis of 2-additivity, requiring $\mu(A) = 0$ whenever $i \notin A$ is equivalent to

\[
m(\{k\}) = 0, \ \forall k \neq i,
\]

\[
m(\{k,l\}) = 0, \ \forall k, l \neq i.
\]

Translating these equalities in the interaction representation leads to

\[
I_{kl} = 0, \ \forall k, l \neq i,
\]

\[
v_k - \frac{1}{2} \sum_{l \neq k} I_{kl} = 0, \ \forall k \neq i.
\]

But last equality reduces to $v_k - \frac{1}{2} I_{ik} = 0$. The inequalities $I_{ik} \geq 0$ have already been shown (equation (22)). For the result concerning favor, simply use property 4.

Remark that $I$ can be replaced by $J$ as well in the above property, since they coincide for 2-additive measures.

It is possible to generalize the concept of veto to several criteria as follows. A group $A \subset X$ of criteria is a veto (resp. a favor) for $\mathcal{H}$ if every criterion in $A$ is a veto (resp. a favor). This leads to the following equation in the case of a veto

\[
\mathcal{H}(t_1, \ldots, t_n) = \bigwedge_{i \in A} t_i \wedge \mathcal{G}(t_1, \ldots, t_n), \quad (23)
\]

and in the case of a favor

\[
\mathcal{H}(t_1, \ldots, t_n) = \bigvee_{i \in A} t_i \vee \mathcal{G}(t_1, \ldots, t_n). \quad (24)
\]

Restricting to the the case of the Choquet integral, we can easily show the following property.
Property 14 For the Choquet integral, a veto effect on a coalition \( A \) of criteria is obtained if and only if the fuzzy measure satisfies \( \mu(B) = 0 \) whenever \( B \not\subseteq A \). Such fuzzy measures are denoted \( \mu^{A^\wedge} \). Similarly, a favor effect on \( A \) is obtained if and only if the fuzzy measure satisfies \( \mu(B) = 1 \) whenever \( B \cap A \neq \emptyset \). Such fuzzy measures are denoted \( \mu^{A^\vee} \). Moreover, \((\mu^{A^\wedge})^* \) has a favor effect for \( A \), and \((\mu^{A^\vee})^* \) a veto effect for \( A \).

Using the above property and the definition of \( I(A) \), it is easy to show that

\[
I^{A^\wedge}(A \cup \{k\}) \geq 0, \quad I^{A^\vee}(A \cup \{k\}) \leq 0, \quad \forall k \not\in A,
\]

where \( I^{A^\wedge}, I^{A^\vee} \) denote respectively the interaction representation of \( \mu^{A^\wedge} \) and \( \mu^{A^\vee} \). As before, the same property holds for the Banzhaf interaction index \( J \).

7. Conclusion

We have shown in this paper how fuzzy measures can afford new tools for decision making, in particular multicriteria decision making. The key concept is the one of representation of a fuzzy measure. It seems that an adequate representation exists for each kind of problem. It is well known that the Möbius representation is the fundamental concept in the Dempster-Shafer theory of evidence, pertaining with decision making under uncertainty. In a similar way, the interaction representation can be said to be fundamental in a multicriteria decision making problem.

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Appendix A

Here are some properties related to the Bernoulli numbers.

Property 1 The coefficients \( \alpha_k \) defined by (11) (Bernoulli numbers) satisfy

(i) \( \alpha_{2k+1} = 0, \quad \forall k > 0, \)

(ii) \( \sum_{j=0}^{k} \binom{k}{j} \alpha_{k-j} = \alpha_k, \quad \forall k > 0, k \neq 1. \)

Proof. (i) this is a fundamental property of the Bernoulli numbers. It can be shown using their exponential generating function

\[
g(x) := \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k,
\]

for every complex number such that \( 0 < |x| < 1 \). Remark that \( g(x) + x/2 \) is an even function. Then, for every \( x \) in the domain,

\[
g(x) + \frac{x}{2} = (g(-x) - \frac{x}{2})
\]
\[
\begin{align*}
    & = 2\alpha_1 x + 2\frac{\alpha_3}{3!} x^3 + \ldots + 2\frac{\alpha_{2k+1}}{(2k+1)!} x^{2k+1} + \ldots + x \\
    & = 2\frac{\alpha_3}{3!} x^3 + \ldots + 2\frac{\alpha_{2k+1}}{(2k+1)!} x^{2k+1} + \ldots \\
    & = 0,
\end{align*}
\]
which implies the result.

(ii) it is equivalent to show that
\[
\sum_{l=1}^{k} \binom{k}{l} \alpha_{k-l} = 0. \tag{A.1}
\]
By definition of \( \alpha_{k-l} \), we have
\[
\alpha_{k-1} = -\sum_{l=0}^{k-2} \frac{\alpha_l}{k-l} \binom{k-1}{l} = -\sum_{i=2}^{k} \frac{\alpha_{k-i}}{i} \binom{k-1}{i}.
\]
Substituting in (A.1) we get
\[
\sum_{l=1}^{k} \binom{k}{l} \alpha_{k-l} = \sum_{l=1}^{k} \left[ \binom{k}{l} - \frac{k}{l} \binom{k-1}{l-1} \right] \alpha_{k-l},
\]
which is clearly 0 since each term in the sum is 0. \(\square\).

**Property 2** The coefficients \( \beta_k \) satisfy the following properties.

(i) \( \beta_0 = \alpha_1 \),

(ii) \( \beta_k = \alpha_k \), \( \forall k \geq 0, k \neq 1 \), \( \beta_1 = -\alpha_1 \),

(iii) the coefficients \( \beta_k \) have the Pascal’s triangle property in the sense that
\[
\beta_k + \beta_{k+1} = \beta_{k+1}, \quad \forall k \leq l,
\]

(iv) \( \beta_k = (-1)^l \beta_{l-k} \),

(v) \( \sum_{k=0}^{l} \beta_k = 0, \quad \forall l > 0 \).

**Proof.**

(i) clear from definition.

(ii) Using first values of the sequence \( \alpha_k \), we obtain easily:
\[
\begin{align*}
\beta_0^0 &= \alpha_0 = 1, \\
\beta_1^1 &= \alpha_0 + \alpha_1 = \frac{1}{2}, \\
\beta_2^2 &= \alpha_0 + 2\alpha_1 + \alpha_2 = \frac{1}{6} = \alpha_2.
\end{align*}
\]
Now let us take a general term $\beta_k^{l}, \ k > 1$. Due to the definition of $\beta_k^{l}$, showing $\beta_k^{l} = \alpha_k$ is equivalent to show that

$$\sum_{l=1}^{k} \binom{k}{l} \alpha_{k-l} = 0.$$ 

But this was already proved (see property 1 (ii)).

(iii) applying the definition of $\beta_k^{l}$ to the left member of the equation gives

$$\sum_{j=0}^{k} \binom{k}{j} \alpha_{l-j} + \sum_{j=0}^{k} \binom{k}{j} \alpha_{l+1-j}$$

$$= \binom{k}{0} \alpha_{l+1} + \sum_{j=0}^{k-1} \left[ \binom{k}{j} + \binom{k}{j+1} \right] \alpha_{l-j} + \binom{k}{k} \alpha_{l-k}$$

$$= \binom{k+1}{0} \alpha_{l+1} + \sum_{j=0}^{k-1} \left[ \binom{k+1}{j+1} \alpha_{l-j} + \binom{k+1}{k} \alpha_{l-k} \right]$$

$$= \binom{k+1}{0} \alpha_{l+1} + \beta_{k+1}^{l+1} - \binom{k+1}{0} \alpha_{l+1} - \binom{k+1}{k} \alpha_{l-k} + \binom{k+1}{k} \alpha_{l-k}$$

$$= \beta_{k+1}^{l+1}.$$

(iv) We show the property by recurrence. It is clearly satisfied for $l = 1$. Let us suppose it is true up to $l > 1$, and verify it is still true for $l + 1$. We suppose first that $l$ is odd. From the hypothesis, (i), (ii) above and Property 1 (ii), we have

$$\beta_k^{l} = -\beta_{k}^{l-1}, \ \forall k \leq l,$$

$$\beta_0^{l} = 0, \ \beta_k^{l} = 0,$$

$$\beta_{l+1}^{l+1} = \alpha_{l+1}, \ \beta_{l+1}^{l+1} = \alpha_{l+1},$$

so that the property is true for $k = 0$. Again we proceed by recurrence to show that $\beta_{k}^{l+1} = \beta_{k+1}^{l+1}$ for $k = 1, \ldots, l$. Let us suppose this is true up to $k$, and verify it is still true for $k + 1$. From (iii) above,

$$\beta_k^{l+1} + \beta_{k+1}^{l+1} = \beta_{k+1}^{l+1},$$

$$\beta_{k-1}^{l+1} + \beta_{k}^{l+1} = \beta_{k+1}^{l+1}.$$  \hspace{1cm} (A.2)

Due to the hypothesis, the second equation can be rewritten as

$$\beta_{k-1}^{l+1} = \beta_{k}^{l+1} + \beta_k^{l},$$

so that by (iii) $\beta_{k+1}^{l} = \beta_{k+1}^{l+1}$. We suppose now that $l$ is even and proceed similarly. We have

$$\beta_k^{l} = \beta_{k-1}^{l}, \ \forall k \leq l,$$

$$\beta_0^{l} = \alpha_1, \ \beta_1^{l} = \alpha_l,$$

$$\beta_0^{l+1} = 0, \ \beta_{l+1}^{l+1} = 0,$$
so that the property is true for \( k = 0 \). We show by recurrence that \( \beta_k^{l+1} = -\beta_{l+1-k}^l \). Let us suppose this is true up to \( k \), and verify it is still true for \( k + 1 \). We have as before (A.2) and (A.3), the second one being rewritten as

\[
\beta_{l-k}^{l+1} = -\beta_{l+1-k}^l.
\]

so that \( \beta_{l-k}^{l+1} = -\beta_{k+1}^{l+1} \). This completes the proof.

(v) Let us consider \( l \) is odd. Then using (iii) above, the result is clear. Now let us consider an even number.

\[
\sum_{k=0}^{2l} \beta_k^{2l} = \sum_{k=0}^{2l} \sum_{j=0}^{k} \binom{k}{j} \alpha_{2l-j} = \sum_{k=0}^{2l} \alpha_k \sum_{j=2l-k}^{2l} \binom{j}{2l-k},
\]

due to property 1. Let us show by recurrence that for every \( k \geq 0 \) and every \( 0 \leq l \leq k \),

\[
\sum_{j=l}^{k} \binom{j}{l} = \binom{k+1}{l+1}.
\]

The relation is trivially satisfied for \( k = 0 \). We suppose it is true at order \( k \), and let us show it is still true at order \( k + 1 \).

\[
\sum_{j=l}^{k+1} \binom{j}{l} = \sum_{j=l}^{k} \binom{j}{l} + \binom{k+1}{l} = \binom{k+1}{l+1} + \binom{k+1}{l} = \frac{(k+1)!}{(l+1)!(k-l)!} + \frac{(k+1)!}{(l+1)!(k+1-l)!} \]

\[
= \frac{(k+1)!(k+1-l) + (k+1)!(l+1)}{(l+1)!(k+1-l)!} = \binom{k+2}{l+1}.
\]

Replacing in the above leads to

\[
\sum_{k=0}^{2l} \beta_k^{2l} = \sum_{k=0}^{l} \alpha_{2k} \left( \frac{2l+1}{2(l-k)+1} \right) = \alpha_{2l+1} = 0,
\]

from property 1 (ii). □.