Generalized Choquet integral on ratio scales

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Abstract

The Choquet integral is widely used in MultiCriteria Decision Making as an aggregation function thanks to its ability to model interaction between criteria. However, we show in an example that it fails to model some simple and usual interaction phenomena. The Cumulative Prospect Theory defined on ratio scales also fails to represent these interaction phenomena. The aim of the paper is to propose an extension of the Choquet integral for ratio scales that is able to model such phenomena. We present the concept of bi-capacity as a generalization of capacities (or fuzzy measures). An axiomatic approach is considered to define the Choquet integral w.r.t. bi-capacities. It is based on information concerning the preferences of the decision maker.

Keywords: multicriteria decision making, Choquet integral, capacities, axiomatic approach.

1 Introduction

Consider a decision problem that depends on $n$ points of views described by attributes $X_1$, \ldots, $X_n$. We wish to model the preferences $\succeq$ of the decision maker (DM) over acts, i.e. elements of $X = X_1 \times \cdots \times X_n$. A classical way is to model $\succeq$ with the help of a utility function $u : X \rightarrow \mathbb{R}$:

$$\forall x, y \in X, \quad x \succeq y \Leftrightarrow u(x) \geq u(y).$$

In measurement theory, it is classical to split the overall evaluation model $u$ into two parts: the utility functions (that map the attributes onto a single satisfaction scale $S$), and the aggregation function (that aggregates values belonging to the commensurate scale $S$):

$$u(x) = F(u_1(x_1), \ldots, u_n(x_n)) \quad \forall x \in X,$$

where $u_i : X_i \rightarrow S$ and $F : S^n \rightarrow S$. The scale $S$ is an interval of $\mathbb{R}$ and depicts the satisfaction degree of the DM.

We are concerned here with the question of which models fit with information concerning the preferences of the decision maker over each attribute, and his preferences about aggregation of criteria (interacting criteria). In [6], this problem was addressed for unipolar scales ($S = \mathbb{R}^+$). It was shown that the conditions induced by these information plus some intuitive conditions lead to a unique possible aggregation operator: the Choquet integral.

The set of all criteria is denoted by $N = \{1, \ldots, n\}$. A capacity $v : \mathcal{P}(N) \rightarrow [0, 1]$ is a set function satisfying $v(\emptyset) = 0$, $v(N) = 1$, and $A \subseteq B$ implies $v(A) \leq v(B)$. The Choquet integral of $u \in \mathbb{R}^n$ w.r.t a capacity $v$ is defined by:

$$C_v(u) = \sum_{i=1}^{n} (u_{\tau(i)} - u_{\tau(i-1)}) v(\{\tau(i), \ldots, \tau(n)\})$$

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where \( u_\tau(0) := 0 \), and \( \tau \) is a permutation such that \( u_\tau(1) \leq \cdots \leq u_\tau(n) \).

In many situations in decision making, it is useful to consider underlying scales \( S \) as bipolar scales, i.e. with a central neutral value (usually 0), considering values above the neutral level as "attractive", and values below it as "repulsive". The motivation for doing so is that human DM do effectively distinguish positive and negative assets, and behave differently. This has lead to models based on the symmetric Choquet integral, or Šipos integral [3], or more generally on CPT (Cumulative Prospect Theory) models in decision under risk or uncertainty [4]. The CPT model is based on two capacities \( v_1, v_2 : \mathbb{V} \rightarrow \mathbb{R} \): \( \text{CPT}(x) = C_{v_1}(x^+) - C_{v_2}(x^-) \), where \( x^+ = x \lor 0 \), and \( x^- = (-x)^+ \).

Despite the ability of these models to cope with many decision behaviors, it is not uncommon to meet practical situations where these models fail to represent preferences, even though these preferences seem rather natural. The rest of this section is devoted to the presentation of such an example.

We consider the problem of evaluating students in a high school according to their levels in mathematics (M), physics (P) and literature (L). The director thinks that scientific subjects are more important than literature, however he would not be satisfied by a student having an important flaw in literature, even though excellent in sciences. As mathematics and physics have some redundancy in skills, the following rules seems to be reasonable and reflect the director’s strategy in decision:

(R1): For a student good at mathematics (M), L is more important than P.
(R2): For a student bad in mathematics (M), P is more important than L.

According to these rules, it is easy to compare students \( A, B, C, D \) whose marks between -10 and 10 (with the neutral value 0) are given in the following table.

<table>
<thead>
<tr>
<th>student</th>
<th>Math.</th>
<th>Physics</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>student A</td>
<td>5</td>
<td>8</td>
<td>-4</td>
</tr>
<tr>
<td>student B</td>
<td>5</td>
<td>6</td>
<td>-2</td>
</tr>
<tr>
<td>student C</td>
<td>-4</td>
<td>3</td>
<td>-8</td>
</tr>
<tr>
<td>student D</td>
<td>-4</td>
<td>1</td>
<td>-6</td>
</tr>
</tbody>
</table>

Clearly, \( A < B \) by Rule R1, and \( C \succ D \) by Rule R2.

Let us try to represent these preferences by the Choquet integral, or a CPT model. An easy calculation shows that the capacity \( v \) in the Choquet integral should satisfy both \( v(\{M, P\}) + v(\{P\}) > 1 \) and \( v(\{M, P\}) + v(\{P\}) < 1 \), which is impossible. On the other hand, considering the CPT model, an easy calculation leads to the following conditions \( v_1(\{P\}) > v_2(\{L\}) \) and \( v_1(\{P\}) < v_2(\{L\}) \), which shows that CPT too is unable to model such preferences. The underlying reason for this is that none of the above models is able to define a specific preference when alternatives have both positive and negative scores.

We propose in this paper an extension of the Choquet integral for ratio scales. It generalizes the Šipos integral as well as the CPT model. It is able to model our basic example.

2 Available information

We aim at modeling a DM preferences on ratio scales. Here, we ask him some special information. This information can be seen as a generalization of the information needed in the Macbeth approach [1]. As in Macbeth, the information is based on the introduction of absolute reference levels over each attribute, and the determination of scales of difference, which, put together, ensures commensurateness. For each attribute \( X_i \), we assume that there exists an element denoted \( 0_i \in X_i \) that is neutral (that is to say neither good nor bad). Values above this element are considered as good or attractive, and form the set \( X_i^+ \). It also
assumed that there exists an element denoted by $1_i \in X_i$ that is considered as *satisfactory*. On the other hand, values below the neutral element are considered as *bad* or *repulsive*, and form the set $X_i^-$. We assume furthermore that there exists an element denoted by $-1_i \in X_i$ that is considered as *unsatisfactory*. It is symmetric to the level $1_i$.

The information asked to the DM is composed of two parts: the preferences of the DM over each attribute (*intra-criterion information*) and his preferences about how to aggregate the criteria (*inter-criteria information*).

### 2.1 Intra-criterion information

The mapping $u_i : X_i \rightarrow \mathbb{R}$ corresponds to the preferences of the DM over attribute $X_i$. However, this particular relation cannot be asked directly to the DM, and $u_i$ shall be deduced from preferences of the DM over general acts of $X$. To this end, generalizing the Macbeth approach [1], we propose to deal with the positive part $X_i^+$ and negative part $X_i^-$ of $X_i$ separately. As a consequence, we introduce the two subsets $X_i^+$ and $X_i^-$ of $X$ defined by

$$X_i^{\pm} := \{(x_i, 0_{-i}) \ ; \ x_i \in X_i^{\pm}\}.$$  

We ask the DM not only the ranking of the elements of $X_i^{\pm}$ but also the difference of satisfaction degree between pairs of elements of $X_i^{\pm}$. Let $u_i^{\pm}$ be defined by:

$$(\text{Intra}_a^+) \ \forall x_i, y_i \in X_i^+, \ u_i^+(x_i) \geq u_i^+(y_i) \Leftrightarrow (x_i, 0_{-i}) \succeq (y_i, 0_{-i}).$$

$$(\text{Intra}_a^-) \ \forall x_i, y_i, z_i, w_i \in X_i^+, \text{ we have } \frac{u_i^+(x_i) - u_i^+(y_i)}{u_i^+(w_i) - u_i^+(z_i)} = k \ (k \in \mathbb{R}^+) \text{ if and only if the difference of satisfaction degree that the DM feels between } (x_i, 0_{-i}) \text{ and } (y_i, 0_{-i}) \text{ is } k \text{ times as large as the difference of satisfaction between } (w_i, 0_{-i}) \text{ and } (z_i, 0_{-i}).$$

$$(\text{Intra}_c^+) \ u_i^+(0_i) = 0 \text{ and } u_i^+(\pm 1_i) = \pm 1.$$ 

$u_i^{\pm}$ defined by (Intra$^+_a$) and (Intra$^+_c$) corresponds to a scale of difference. Any scale of difference is given up to two degrees of freedom. Condition (Intra$^+_c$) fixes these two degrees of freedom. Henceforth, $u_i^{\pm}$ is uniquely determined by (Intra$^+_a$), (Intra$^+_c$) and (Intra$^+_c$). The scale $u_i$ is then defined by $u_i(x_i) = u_i^+(x_i)$ if $x_i \in X_i^+$ and $u_i(x_i) = u_i^-(x_i)$ otherwise.

### 2.2 Inter-criteria information

Considering two acts $x, y \in X$ and $A \subseteq N$, we use the notation $(x_A, y_{-A})$ to denote the act $z \in X$ such that $z_i = x_i$ if $i \in A$ and $y_i$ otherwise.

For a usual capacity $v$, we have $C_v(1_A, 0_{-A}) = v(A)$. Hence $v(A)$ represents the overall score of the binary alternative $(1_A, 0_{-A})$ which is “satisfactory” on criteria in $A$, and “neutral” for all other criteria. In order to generalize the notion of capacity, the idea is to consider ternary alternatives of the kind $(1_A, -1_{A'}, 0_{-A \cup A'})$ mixing neutral elements, satisfactory elements with unsatisfactory ones. The following subset of $X$ depicts the set of all possible ternary acts:

$$X_{\{1,0,1\}} := \{(1_A, -1_{A'}, 0_{-A \cup A'}) \ ; \ (A, A') \in Q(N)\},$$

where $Q(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N)|A \cap B = \emptyset\}$. As for intra-criterion information, let $\mu$ be defined on $Q(N)$ by (for all $(A, A'), (B, B'), (C, C'), (D, D') \in Q(N)$): 

$$(\text{Inter}_a) \ \mu(A, A') \geq \mu(B, B') \Leftrightarrow (1_A, -1_{A'}, 0_{-A \cup A'}) \geq (1_B, -1_{B'}, 0_{-B \cup B'}).$$

$$(\text{Inter}_b) \ \frac{\mu(A, A') - \mu(B, B')}{\mu(C, C') - \mu(D, D')} = k \ (k \in \mathbb{R}^+) \text{ if and only if the difference of satisfaction degree that the DM feels between } (1_A, -1_{A'}, 0_{-A \cup A'}) \text{ and } (1_B, -1_{B'}, 0_{-B \cup B'}) \text{ is } k \text{ times as large as the difference of satisfaction between } (1_C, -1_{C'}, 0_{-C \cup C'}) \text{ and } (1_D, -1_{D'}, 0_{-D \cup D'}).$$

3
(Inter\(_c\)) \( \mu(\emptyset, \emptyset) = 0 \), \( \mu(N, \emptyset) = 1 \) and \( \mu(\emptyset, N) = -1 \).

\( \mu \) is uniquely determined by (Inter\(_a\)), (Inter\(_b\)) and (Inter\(_c\)).

Since \( \mu(A, B) \) is closely related to the overall evaluation of the act \((\mathbf{1}_A, -\mathbf{1}_B, \mathbf{0}_{-A \cup B})\), monotonicity of the aggregation function implies some monotonicity conditions on \( \mu : A \subset A' \Rightarrow \mu(A, B) \leq \mu(A', B) \), and \( B \subset B' \Rightarrow \mu(A, B) \geq \mu(A, B') \). We call bi-capacities such functions satisfying (Inter\(_c\)). It is similar to the concept of bi-cooperative games in Game Theory [2].

### 3 Determination of the model

Specifying (2), it is now natural to write \( u \) as \( u(x) = F_\mu(u_1(x_1), \ldots, u_n(x_n)) \), where \( F_\mu \) is the aggregation operator. \( F_\mu \) depends on \( \mu \) in a way that is not known for the moment.

#### 3.1 Measurement conditions

We study the conditions on \( F \) implied by the information asked in Section 2. We introduce the following axioms.

\[ (\text{LM}) : \] For any bi-capacities \( \mu, \mu' \) on \( Q(N) \), for all \( x \in \mathbb{R}^n \) and \( \gamma, \delta \in \mathbb{R} \),
\[ F_{\gamma \mu + \delta \mu'}(x) = \gamma F_\mu(x) + \delta F_{\mu'}(x) \]

\[ (\text{In}) : \] For any bi-capacity \( \mu \) on \( Q(N) \), \( \forall x, x' \in \mathbb{R}^n \),
\[ x_i \leq x'_i, \forall i \in N \Rightarrow F_\mu(x) \leq F_\mu(x') \]

\[ (\text{PW}) : \] For any bi-capacity \( \mu, F_\mu(1_A, -1_{A'}, 0_{-A \cup A'}) = \mu(A, A'), \forall (A, A') \in Q(N) \).

\[ (\text{weak SPL}^+) : \] For any bi-capacity \( v \) on \( Q(N) \), for all \( A, C \subset N \), \( \alpha > 0 \), and \( \beta \geq 0 \),
\[ F_\mu((\alpha + \beta)_A, \beta_{-A}) = \alpha F_v(1_A, 0_{-A}) + \beta v(N, \emptyset) \]

These axioms are basically deduced from the measurement conditions on \( u_i^\pm \) and \( \mu \). By conditions (Inter\(_a\)) and (Inter\(_b\)), \( u(1_A, -1_{A'}, 0_{-A \cup A'}) \) and \( \mu(A, A') \) correspond to two possible satisfaction scales related to the act \((1_A, -1_{A'}, 0_{-A \cup A'}) \in X \setminus \{-1, 0, 1\} \). By conditions (Intra\(_a^\pm\)) and (Intra\(_b^\pm\)), \( u(x_i, 0_{-i}) \) and \( u_i^+(x_i) \) correspond to two possible scales of difference related to the same act \((x_i, 0_{-i}) \in X \setminus \{0\} \). From the measurement conditions implied by scales of difference, requirements (PW) and (weak SPL\(^+\)) are shown [7, 6]. (weak SPL\(^+\)) is a weak version of axiom SPL (invariance to shift and linear transformation) proposed by Marichal [8] in order to characterize the usual Choquet integral. On ratio scales, the 0 should not be shifted. However, when only attractive elements are considered, the real meaning of 0 is lost and it can be shifted. This gives an interpretation of (weak SPL\(^+\)). Moreover, (In) is a necessary requirement for an aggregation function. Finally, (LM) is partially deduced from the measurement conditions [7, 6], and can be seen as natural since it is satisfied by the usual Choquet integral for a capacity.

For \( A \subset N \), consider the following application \( \Pi_A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) defined by \( (\Pi_A(x))_i = x_i \) if \( i \in A \) and \(-x_i \) otherwise. By (PW), \( \mu(B, B') \) corresponds to the point \((1_B, -1_{B'}, 0_{B \cup B'}) \). Define \( \Pi_A \circ \mu(B, B') \) as the term of the bi-capacity associated to the point \( \Pi_A(1_B, -1_{B'}, 0_{B \cup B'}) = (1_{(B \cap A) \cup (B' \setminus A)}, -1_{(B \cup A) \cup (B' \setminus A)}, 0_{-B \cup B'}) \). Hence we set
\[ \Pi_A \circ \mu(B, B') := \mu((B \cap A) \cup (B' \setminus A), (B \setminus A) \cup (B' \cap A)) \]

By symmetry arguments, it is reasonable to have \( F_{\Pi_A \circ v}(\Pi_A(x)) \) being equal to \( F_v(x) \).

\[ (\text{Sym}) : \] For any \( v : Q(N) \rightarrow \mathbb{R} \), we have for all \( A \subset N \)
\[ F_v(x) = F_{\Pi_A \circ v}(\Pi_A(x)) \]
3.2 Expression of $F_v$

The following result can be shown.

**Theorem 1** \{ $F_v$ \} satisfies (LM), (In), (PW), (weak SPL$^+$) and (Sym) if and only if for any bi-capacity $v$, and for any $N^+ \subset N$, $x \in \Sigma_{N^+}$,

$$F_v(x) = C_{\nu_{N^+}}(x_{N^+}, -x_{-N^+})$$

where $\nu_{N^+}(C) := v(C \cap N^+, C \cap (N \setminus N^+))$, and $\Sigma_{N^+} := \{ x \in \mathbb{R}^n, x_{N^+} \geq 0, x_{N \setminus N^+} < 0 \}$.

The following result can be shown.

**Proposition 1** If $v$ is a bi-capacity of the CPT type, then $F_v$ coincide with a CPT model. Moreover, if $v$ is a symmetric bi-capacity, then $F_v$ coincides with the Šipoš integral.

3.3 Application of bi-capacities to the example

Let us come back to our example, unsolved with the CPT model. Applying the definition of the generalized Choquet integral, the two preferences $A \prec B$ and $C \succ D$ implies that $v(\{M, P\} \setminus \emptyset) - v(\{M, P\}, \{L\}) > v(\{P\}, \emptyset)$ and $-v(\emptyset, \{L\}) < v(\{P\}, \{M, L\}) - v(\emptyset, \{M, L\})$. There is no contradiction between these two inequalities, and so the preference can be properly represented.

References


