Set functions over finite sets: transformations and integrals

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1 Introduction

Measures in a finite setting are positive valued set functions with some characteristic properties (e.g. additivity for classical measures). Although the finiteness of the universe considerably restricts the interest of the measure concept, it also allows other viewpoints, coming from various fields of mathematics, as combinatorics, game theory, complexity, etc., when viewing measures as particular set functions or pseudo-Boolean functions.

We intend to illustrate this viewpoint into this chapter. We will introduce various transformations of set functions, and the notion of integral will appear naturally as an extension of set functions. In a last part, we will consider set functions valued on linearly ordered sets, and corresponding transformations and integrals.

Throughout the paper, we assume a finite space \( N \) with \( n \) elements, denoted simply \( 1, 2, \ldots, n \) if there is no fear of ambiguity. In a similar way, \( s, t, \ldots \) will denote the cardinality of subsets \( S, T, \ldots \) of \( N \). We denote by \( \land, \lor \) the minimum and maximum operators on \( \mathbb{R} \).

2 Set functions over finite sets

We consider real valued set functions \( v : \mathcal{P}(N) \rightarrow \mathbb{R} \), and several particular cases. Set functions vanishing on the empty set are called games, while capacities [3] or fuzzy measures [31], which we will always denote by \( \mu \), refer to games which are monotonic with respect to inclusion, i.e.

\[
A \subset B \Rightarrow \mu(A) \leq \mu(B).
\]

Consequently, fuzzy measures assume only positive values. In applications, it is often required in addition that \( \mu(N) = 1 \).
For any set function $v$, the dual set function or conjugate set function of $v$ is defined by

$$
\varphi(S) := v(N) - v(S^c), \quad \forall S \subseteq N,
$$

(1)

where $S^c$ is the complement set of $S$.

A particular family of set functions of interest are the unanimity games. For any $S \subseteq N$, the unanimity game w.r.t. $S$ is defined by:

$$
u_S(T) := \begin{cases} 
1, & \text{if } T \supset S \\
0, & \text{otherwise}.
\end{cases}
$$

(2)

Another view of set functions which has its own interest is given by pseudo-Boolean functions. We introduce briefly the topic here, more can be found in [17, 12].

Any function $f : \{0,1\}^n \rightarrow \mathbb{R}$ is said to be a pseudo-Boolean function. By making the usual bijection between $\{0,1\}^n$ and $\mathcal{P}(N)$, pseudo-Boolean functions on $\{0,1\}^n$ coincide with real-valued set functions on $N$. It has been shown by Hammer and Rudeanu [20] that any pseudo-Boolean function can be written in a multilinear form:

$$
f(x) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \quad \forall x \in \{0,1\}^n.
$$

(3)

The coefficients $a(T)$ will be explained below. The monomials $\prod_{i \in T} x_i$ correspond to unanimity games $\nu_T$. In terms of game theory, equation (3) gives the decomposition of a game on the basis of unanimity games.

Note that (3) can be put in an equivalent form, which is

$$
f(x) = \sum_{T \subseteq N} a(T) \bigwedge_{i \in T} x_i, \quad \forall x \in \{0,1\}^n.
$$

(4)

A topic of interest concerns the extension of pseudo-Boolean functions to $[0,1]^n$. An obvious way to do this is to extend expressions (3) and (4) to the whole hypercube $[0,1]^n$. The first one is called the multilinear extension, while the second is the Lovász extension.

The multilinear extension of $f$, given by

$$
\hat{f}(x) := \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \quad \forall x \in [0,1]^n,
$$

(5)

is the only multilinear function which extends $f$, hence its name (see Owen [26]). It performs the classical linear interpolation of $f$ in $[0,1]^n$.

The Lovász extension of $f$ [21, 30] is given by

$$
\tilde{f}(x) := \sum_{T \subseteq N} a(T) \bigwedge_{i \in T} x_i, \quad \forall x \in [0,1]^n.
$$

(6)
Defining the simplex $B_n = \{x \in [0, 1]^n | x_{\pi(1)} \leq \cdots \leq x_{\pi(n)} \}$ where $\pi$ is a permutation on $N$, the Lovász extension is the unique affine function which interpolates $f$ between the $n + 1$ vertices of $B_n$. By contrast, the multilinear extension interpolates between all vertices.

As a prelude to the next section, we introduce informally some linear transforms over set functions.

**Möbius transform**: in combinatorics, the Möbius transform is well-known (see e.g. Rota [27]). For any set function $v$, its Möbius transform $m^v$ is defined by:

$$m^v(S) := \sum_{T \subseteq S} (-1)^{|S| - |T|}v(T), \quad \forall S \subseteq N. \quad (7)$$

The inverse transform is the Zeta transform, expressed by:

$$v(S) = \sum_{T \subseteq S} m^v(T), \quad \forall S \subseteq N. \quad (8)$$

$m^v$ is called the Möbius representation of $v$. In game theory, this corresponds to the dividends of a game. Comparing (8) and (3), we see that $m^v(T)$ is nothing else than $a(T)$. In other words, the Möbius transform is the coordinates of $\mu$ in the basis of unanimity games. In Dempster-Shafer theory of evidence, $m^v$ is called the basic probability mass assignment [28].

**interaction transform**: it has been proposed by Grabisch [7], and is defined for any set function $v$ by:

$$I^v(S) := \sum_{T \subseteq N \setminus S} \frac{(n - t - s)!}{(n - s + 1)!} \sum_{K \subseteq S} (-1)^{|S| - k} v(K \cup T), \forall S \subseteq N. \quad (9)$$

This definition extends in fact the Shapley value $\phi^v$ [29] and the interaction index $I_{ij}$ for a pair of elements $i, j$ in $N$, introduced by Murofushi and Soneda [24]. They are defined by

$$\phi^v(i) := \sum_{S \subseteq N \setminus i} \frac{(n - s - 1)!s!}{n!} [v(S \cup \{i\}) - v(S)], \forall i \in N \quad (10)$$

$$I_{ij} := \sum_{S \subseteq N \setminus \{i, j\}} \frac{(n - s - 2)!s!}{(n - 1)!} [v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)], \quad (11)$$

We have $I^v(\{i\}) = \phi^v(i)$ and $I^v(\{i, j\}) = I_{ij}$. $I^v$ is often referred as the interaction index.

The interaction index has a natural interpretation in the framework of cooperative game theory and multicriteria decision making. It has been axiomatized by Grabisch and Roubens [19].

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**co-Möbius transform:** for any set function \( v \), it is defined by

\[
\hat{v}(S) := \sum_{T \subseteq N \setminus S} (-1)^{|T|} v(T) = \sum_{T \subseteq S} (-1)^{|T|} v(N \setminus T), \forall S \subset N. \quad (12)
\]

\( \hat{v} \) corresponds to the *commonality function* of Shafer [28]. The analogy with the Möbius transform can be noticed.

### 3 Transformations of set functions

We introduce formally the concept of transformation of a set function, as proposed by Denneberg and Grabisch [6], borrowing the formalism of transformation and operators used in combinatorics (see Berge [1]).

An *operator* is a two-place set function \( \Phi : \mathcal{P}(N) \times \mathcal{P}(N) \rightarrow \mathbb{R} \). The multiplication \( \star \) between operators and set functions is defined as follows, for every \( A, B, C \subset N \)

\[
(\Phi \star \Psi)(A, B) := \sum_{C \subseteq N} \Phi(A, C) \Psi(C, B),
\]

\[
(\Phi \star v)(A) := \sum_{C \subseteq N} \Phi(A, C) v(C),
\]

\[
(v \star \Psi)(B) := \sum_{C \subseteq N} v(C) \Psi(C, B).
\]

The Kronecker’s delta

\[
\Delta(A, B) := \begin{cases} 
1 & \text{if } A = B \\
0 & \text{else} 
\end{cases}
\]

is the unique neutral element from the left and from the right. If it exists, the inverse of \( \Phi \) is denoted \( \Phi^{-1} \), satisfying \( \Phi \star \Phi^{-1} = \Delta \), \( \Phi^{-1} \star \Phi = \Delta \).

It can be shown that the family

\[
\mathcal{G} := \{ \Phi : \mathcal{P}(N) \times \mathcal{P}(N) \rightarrow \mathbb{R} \mid \Phi(A, A) = 1 \forall A \subset N, \Phi(A, B) = 0 \text{ if } A \not\subset B \}
\]

of functions of two variables together with the operation \( \star \) forms a group, and the inverse \( \Phi^{-1} \in \mathcal{G} \) of \( \Phi \in \mathcal{G} \) computes recursively through

\[
\Phi^{-1}(A, A) = 1,
\]

\[
\Phi^{-1}(A, B) = - \sum_{A \subset C \subseteq \complement B} \Phi^{-1}(A, C) \Phi(C, B) \quad \text{if } A \not\subset B.
\]

The *Zeta operator* \( Z(A, B) \), defined by

\[
Z(A, B) := \begin{cases} 
1 & \text{if } A \subset B \\
0 & \text{else}
\end{cases}
\]
and its inverse the \textit{M"{o}bius operator} correspond to our previous definitions, i.e. with former notations,

\[ m = v \star Z^{-1}, \quad v = m \star Z. \]  

(13)

The next fundamental operator to introduce is the so-called \textit{inverse Bernoulli operator} $\Gamma$:

\[ \Gamma(A, B) := \begin{cases} \frac{1}{|B \setminus A| + 1} & \text{if } A \subset B, \\ 0 & \text{else} \end{cases} . \]

The interaction index is recovered by

\[ I = \Gamma \star m. \]  

(14)

We turn now to a special class of operators, satisfying

\[ \Phi(A, B) = \Phi(\emptyset, B \setminus A) \quad \text{for } A \subset B, \]  

(15)
i.e. they can be represented by an ordinary set function $\varphi(A) := \Phi(\emptyset, A)$, denoted with the corresponding small greek letter. In fact, the set of such operations forms an Abelian group, as well as the corresponding set of set functions:

\[ g := \{ \varphi : \mathcal{P}(N) \to \mathbb{R} \mid \varphi(\emptyset) = 1 \} \]

with operation $\star$ defined by

\[ \varphi \star \psi(A) := \sum_{C \subset A} \varphi(C)\psi(A \setminus C), \quad A \subset N. \]

The neutral element $\delta$ of $g$ is

\[ \delta(A) := \begin{cases} 1 & \text{if } A = \emptyset, \\ 0 & \text{else} \end{cases} , \]

and the inverse of $\varphi$ is denoted $\varphi^{-1}$. Since $Z$ and $\Gamma$ have property (15), we can introduce the corresponding Zeta function and Bernoulli function:

\[ \zeta(A) = 1 \quad \text{for all } A \in \mathcal{P}, \]

\[ \gamma(A) = \frac{1}{|A| + 1}, \quad A \in \mathcal{P}. \]

If moreover $\varphi$ is a function only of the cardinal of sets, then we call it a \textit{cardinality function}, and the corresponding $\Phi$ a \textit{cardinality operator}. Note that $Z$ and $\Gamma$ have also this property.

There is a general formula for the inverse of cardinality operators (which is also cardinal). More specifically, if $\varphi(A) = f(|A|)$, then $\varphi^{-1}(A) = f^{-1}(|A|)$, with $f^{-1}$ defined recursively by

\[ f^{-1}(0) := 1, \]

\[ f^{-1}(m) := -\sum_{k=0}^{m-1} \binom{m}{k} f(m-k) f^{-1}(k), \quad m \in \mathbb{N}. \]  

(16)
With this formula, we get $\zeta^{s-1}$ the Möbius function, and $\gamma^{s-1}$ the Bernoulli function, giving rise to the Bernoulli numbers.

Coming back to the interaction index, we have

$$I = \Gamma \ast m = \Gamma \ast (v \ast Z^{-1}),$$

which can be rearranged to obtain $I = v \ast I$. $I$ is called the interaction operator, and is not in the group $G$.

This formalism permits to compute easily, simply by combining and inverting operators, all formulas between the different representations of set functions (Möbius transform, interaction index, etc.). These formulas are summarized in subsequent tables. We introduce some notations. The Bernoulli numbers $B_k$, are defined by:

$$B_k := -\sum_{l=0}^{k-1} \frac{B_l}{k-l+1} \binom{k}{l}, \; k > 0,$$

and $B_0 = 1$. First numbers of the sequence are $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, etc., and $B_{2k+1} = 0$ for all $k > 0$. The numbers $\beta^{\ell}_k$ are defined from the Bernoulli numbers by

$$\beta^{\ell}_k := \sum_{j=0}^{k} \binom{k}{j} B_{l-j}, \; k,\ell = 0, 1, 2, \ldots$$

First values of $\beta^{\ell}_k$ are

$$\begin{array}{c|cccc}
  k \backslash \ell & 0 & 1 & 2 & 3 \\
  \hline
  0 & 1 & -\frac{1}{2} & \frac{1}{6} & 0 & -\frac{1}{30} \\
  1 & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{30} \\
  2 & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} & -\frac{1}{30} \\
  3 & 0 & -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{30} \\
  4 & & & & -\frac{1}{30} \\
\end{array}$$

This table has a property similar to the Pascal triangle, i.e. one coefficient $\beta^{\ell+1}_{k+1}$ is the sum of the two above $\beta^{\ell}_k + \beta^{\ell+1}_k$. For other properties, see [8].
\[ m(S) = \sum_{T \supset N \setminus S} (-1)^{|S|} \rho(T) \]
\[ m = \sum_{T \supset S} (\rho(T) - \sum_{T \supset S} (-1)^{|T|}I(T)) \cdot m(T) \]

<table>
<thead>
<tr>
<th>$v$</th>
<th>$m$</th>
<th>$I$</th>
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<tbody>
<tr>
<td>$m(S)$ = $\sum_{T \supset N \setminus S} (-1)^{</td>
<td>S</td>
<td>}v(T)$</td>
</tr>
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\[
\begin{align*}
 v(S) &= \sum_{T \subset N \setminus S} (-1)^{|S|} \bar{m}(T) \\
m(S) &= \sum_{T \supset S} (-1)^{|T|-s} \bar{m}(T) \\
I(S) &= \sum_{T \supset S} (-1)^{|T|-s} \frac{1}{t-s+1} \bar{m}(T)
\end{align*}
\]

Table 2: Tables of conversion between $\bar{m}$ and other representations

Another formalization is based on matrices, putting the coefficients $v(S)$ for all $S \subset N$ into a vector. Then, transformations leads to peculiar forms of matrices (see [16]).

We give some properties of the above introduced transforms (for more details, see [10, 12, 6]).

**Property 1** For any set function $v$ and its conjugate $\bar{v}$, we have
\[
\begin{align*}
 \bar{m}^v(S) &= (-1)^{|S|+1} m^v(S), \quad \forall S \subset N, S \neq \emptyset \\
m^v(S) &= (-1)^{|S|+1} \sum_{T \supset S} m^v(T), \quad \forall S \subset N, S \neq \emptyset.
\end{align*}
\]

**Property 2** Any set function can be decomposed on the set of unanimity games or their conjugate:
\[
\begin{align*}
 v(S) &= \sum_{T \subset N} m^v(T) u_T(S) \\
v(S) &= \sum_{T \subset N, T \neq \emptyset} (-1)^{|T|+1} \bar{m}^v(T) \bar{u}_T(S) \\
v(S) &= \sum_{T \subset N} m^v(T) \bar{u}_T(S)
\end{align*}
\]

**Property 3** For any set function $v$ and its conjugate $\bar{v}$, we have
\[
\begin{align*}
 I^v(\emptyset) &= v(N) - I^v(\emptyset) \\
 I^v(S) &= -(-1)^{|S|} I^v(S), \quad \forall S \subset N, S \neq \emptyset.
\end{align*}
\]

**Property 4** Let $\mu$ be a fuzzy measure with $\mu(N) = 1$. Then the interaction index $I_{ij}$ ranges in $[-1, +1]$. $I_{ij} = 1$ if and only if $\mu = u_{\{i,j\}}$, the unanimity game for the pair $i, j$. Similarly, $I_{ij} = -1$ if and only if $\mu = a_{\{i,j\}}$, the dual measure of the unanimity game for the pair $i, j$. 

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Property 5 [2] A set of $2^n$ coefficients $m(S)$, $S \subseteq N$ corresponds to the Möbius transform of a fuzzy measure if and only if

(i) $m(\emptyset) = 0$, $\sum_{S \subseteq N} m(S) = 1$,

(ii) $\sum_{i \in T \subseteq S} m(T) \geq 0$, for all $S \subseteq N$, for all $i \in S$.

Property 6 [10] A set of $2^n$ coefficients $I(S)$, $S \subseteq N$ corresponds to the interaction transform of a fuzzy measure if and only if

(i) $\sum_{S \subseteq N} B_i I(S) = 0$,

(ii) $\sum_{i \in N} I(\{i\}) = 1$,

(iii) $\sum_{S \subseteq N \setminus i} \beta^{|S|} I(S \cup \{i\}) \geq 0$, $\forall i \in N$, $\forall T \subseteq N \setminus \{i\}$.

4 The Choquet and Šipoš integrals

Definition 1 Let $f$ be a positive real-valued function on $N$, and $v$ a set function. Let us denote $f(i)$ by $f_i$, for every $i$ in $N$, thus considering $f$ as an element of $(\mathbb{R}^+)^n$. Then the Choquet integral of $f$ with respect to $v$ is given by

$$C_v(f) := \sum_{i=1}^n [f(i) - f(i-1)]v(A_i),$$

where $^{(i)}$ indicates a permutation on $N$ so that $f(1) \leq f(2) \leq \cdots \leq f(n)$, and $A_i := \{(i), \ldots, (n)\}$. Also $f_{(0)} := 0$.

It has been shown by Chateauneuf and Jaffray [2], extending Dempster [4] (see also Walley [33]), that the Choquet integral of positive integrands can be expressed using the Möbius transform:

$$C_v(f) = \sum_{S \subseteq N} m^v(S) \cdot \bigwedge_{i \in S} f_i$$

We recognize here the Lovász extension of $v$ (see (6)), which shows clearly the link between the set function and the Choquet integral: the latter is an extension of the former, which means that for any $S \subseteq N$, if $f^S$ is defined as $f^S_i = 1$ whenever $i \in S$, and 0 otherwise, then $C_v(f^S) = v(S)$.

The next step is to define the Choquet integral for real-valued integrands. For any $f \in \mathbb{R}^n$, we introduce $f^+ := f \vee 0$, $f^- := (-f) \vee 0$ the positive and negative parts of $f$, i.e. such that $f = f^+ - f^-$. Two definitions exist [5]:

$$C_v(f)$$

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• the *asymmetric integral*, which is the usual definition of Choquet integral:

\[ C_v(f) := C_v(f^+) - C_v(f^-), \] (26)

• the *symmetric integral*, which is also called the Šipoš integral [32]:

\[ \hat{C}_v(f) := C_v(f^+) - C_v(f^-). \] (27)

They satisfy for all \( f \in (\mathbb{R}^+)^n \), \( C_v(-f) = -C_v(f) \) and \( \hat{C}_v(-f) = -\hat{C}_v(f) \), hence their names. The explicit expression of the Šipoš integral is, for any \( f \in \mathbb{R}^n \):

\[
\hat{C}_v(f) = \sum_{i=1}^{p-1} (f(i) - f(i+1)) v(\{(1), \ldots, (i)\}) \\
+ f(p) v(\{(1), \ldots, (p)\}) \\
+ f(p+1) v(\{(p+1), \ldots, (n)\}) \\
+ \sum_{i=p+2}^{n} (f(i) - f(i-1)) v(\{(i), \ldots, (n)\})
\] (28)

with \( f(1) \leq \cdots \leq f(p) < 0 \leq f(p+1) \leq \cdots \leq f(n) \). By contrast, expression (24) of the Choquet integral does not change when \( f \in \mathbb{R}^n \). This means that the Šipoš integral behaves differently on positive and negative numbers (true zero, ratio scale), while the Choquet integral does not differentiate them (arbitrary position of the zero, difference scale)(more on this topic and its relation to decision making in [15]).

We turn to the expression of these two integrals using the various transformations introduced above.

**Property 7** Let \( v \) be a set function, \( m, m, I \) their Möbius, co-Möbius and interaction transforms, and \( f \) a real-valued function on \( N \). Then the Choquet integral of \( f \) w.r.t. \( v \) is expressed by:

\[
C_v(f) = \sum_{S \subseteq N} m(S) \bigcap_{i \in S} f_i
\] (29)

\[
C_v(f) = \sum_{S \subseteq N, S \neq \emptyset} (-1)^{s+1} m(S) \bigvee_{i \in S} f_i
\] (30)

\[
C_v(f) = \sum_{S \subseteq N} \left[ \sum_{T \supseteq S} B_{-i} I^+(T) \right] \bigcap_{i \in S} f_i + \\
\sum_{S \subseteq N, S \neq \emptyset} (-1)^{s+1} \left[ \sum_{T \supseteq S} B_{-i} I^-(T) \right] \bigvee_{i \in S} f_i
\] (31)

with \( I^+ \) indicating a restriction so that only terms with positive interaction are taken into account, and similarly for \( I^- \).
Property 8 Let \( v \) be a set function, \( m, \hat{m}, I \) their M"obius, co-M"obius and interaction representations, and \( f \) a real-valued function on \( N \). Then the Šipoš integral of \( f \) w.r.t. \( v \) is expressed by:

\[
\hat{c}_v(f) = \sum_{A \subseteq N} m(A) \left[ \bigwedge_{i \in A} f_i^+ - \bigwedge_{i \in A} f_i^- \right]
\]

\[
= \sum_{A \subseteq N^+} m(A) \bigwedge_{i \in A} f_i^+ + \sum_{A \subseteq N^-} m(A) \bigvee_{i \in A} f_i^-
\]

\[
\hat{c}_v(f) = \sum_{A \subseteq N^+} (-1)^{|A|+1} \hat{m}^v(A) \bigvee_{i \in A} f_i^+ + \sum_{A \subseteq N^-} (-1)^{|A|+1} \hat{m}^v(A) \bigwedge_{i \in A} f_i^-
\]

\[
\hat{c}_v(f) = \sum_{A \subseteq N^+} \left( \sum_{B \subseteq N \setminus A} B_{|B|} I^+(A \cup B) \right) \bigwedge_{i \in A} f_i^+
\]

\[
+ \sum_{A \subseteq N^-} \left( \sum_{B \subseteq N \setminus A} B_{|B|} I^+(A \cup B) \right) \bigvee_{i \in A} f_i^-
\]

\[
+ \sum_{A \subseteq N^+ \neq \emptyset} (-1)^{|A|+1} \left( \sum_{B \subseteq N \setminus A} B_{|B|} I^-(A \cup B) \right) \bigvee_{i \in A} f_i^-
\]

\[
+ \sum_{A \subseteq N^- \neq \emptyset} (-1)^{|A|+1} \left( \sum_{B \subseteq N \setminus A} B_{|B|} I^-(A \cup B) \right) \bigwedge_{i \in A} f_i^-, \tag{35}
\]

with \( N^+ := \{ i \in N | f_i \geq 0 \} \), and \( N^- := N \setminus N^+ \).

5 \( k \)-additive measures

In this section, we focus on capacities (or fuzzy measures), although the concept of \( k \)-additivity can be defined for any set function.

\( k \)-order additive measures or \( k \)-additive measures for short have been introduced by Grabisch in an attempt to decrease the exponential complexity of fuzzy measures in practical applications, since a fuzzy measure defined on a set of \( n \) elements requires \( 2^n \) real coefficients for its definition. A means which has been often used for this is to introduce the property of decomposability: a fuzzy measure \( \mu \) is decomposable if the measure of any subset can be expressed as a function of the measures of each element in the set. Thus we need only to define the distribution of \( \mu \) over \( N \), hence \( n \) coefficients instead of \( 2^n \). The most usual example in this category are additive measures. But it appears that this is too drastic a simplification, which is too limitative,
especially in multiattribute decision making. One can think of a distribution
defined not only for singletons, but also for pairs. An adequate way to define
this is to refer to pseudo-Boolean functions, and especially their multilinear
form (3). An additive measure (defined by a distribution on singletons) has
a linear expression \( f(x) = \sum a_i x_i \), and the coefficients \( a_i \)'s (i.e. the Möbius
transform) are indeed the distribution itself. Then a fuzzy measure of which
the multilinear extension is limited to a degree 2, (or 3, etc.) can be ex-
pressed only by a distribution on singletons and pairs (or also on triples,
etc.). Equation (3) will then provide the value of \( \mu \) for every subset.

**Definition 2** A fuzzy measure \( \mu \) is said to be \( k \)-additive if its Möbius transform satisfies \( m(S) = 0 \) for any \( S \) such that \( s > k \), and there exists at least
one subset \( S \) of \( N \) of exactly \( k \) elements such that \( m(S) \neq 0 \).

The following property of \( k \)-additive measures is fundamental.

**Property 9** Let \( \mu \) be a \( k \)-additive measure on \( N \). Then

(i) \( I(S) = \hat{m}(S) = 0 \) for every \( S \subset N \) such that \( |S| > k \),

(ii) \( I(S) = m(S) = \hat{m}(S) \) for every \( S \subset N \) such that \( |S| = k \).

Thus, \( k \)-additive measures can be represented by a limited set of coefficients,
either \( m(S) \), \( S \leq k \), or \( I(S) \), \( S \leq k \), or equivalently \( \hat{m}(S) \), \( |S| \leq k \), i.e. at
most \( \sum_{i=1}^{k} \binom{n}{i} \) coefficients.

The case of 2-additive measures is particularly appealing, since it is much
more general than additive measures, while remaining of low complexity. We
detail properties of this case in the sequel. A first fact is that any 2-additive
measure \( \mu \) can be expressed only by \( \mu(\{i\}) \), \( \mu(\{i,j\}) \), for all \( i, j \in N \), hence
the notion of “distribution” for singletons and pairs suggested above.

**Property 10** Any 2-additive measure can be written as:

\[
\mu(A) = \sum_{\{i,j\} \in A} \mu(\{i,j\}) - (|A| - 2) \sum_{i \in A} \mu(\{i\}), \quad \forall A \subset N, |A| \geq 2 \quad (36)
\]

Moreover, the interaction index \( I_{ij} \) reduces to:

\[
I_{ij} = \mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\}). \quad (37)
\]

The Choquet and Šipoš integrals have a particularly interesting expression
for 2-additive measure, in terms of the interaction transform.
Property 11 Let \( f \in \mathbb{R}^n \) and \( \mu \) a 2-additive measure. Then:

\[
\mathcal{C}_\mu(f) = \sum_{i,j} (f_i \land f_j) I_{i,j} + \sum_{i,j} (f_i \lor f_j) |I_{i,j}| + \sum_{i=1}^{n} f_i (\phi_i - \frac{1}{2} \sum_{j \neq i} |I_{i,j}|) \quad (38)
\]

\[
\hat{\mathcal{C}}_\mu(f) = \sum_{i,j \in N^+, I_{i,j} > 0} (f_i \land f_j) I_{i,j} + \sum_{i,j \in N^-, I_{i,j} < 0} (f_i \lor f_j) I_{i,j}
\]

\[+ \sum_{i,j \in N^+, I_{i,j} > 0} (f_i \lor f_j) |I_{i,j}| + \sum_{i,j \in N^-, I_{i,j} < 0} (f_i \land f_j) |I_{i,j}|
\]

\[+ 2 \sum_{i \in N^+} f_i \left( \sum_{j \in N^-} |I_{i,j}| \right) + 2 \sum_{i \in N^-} f_i \left( \sum_{j \in N^+} |I_{i,j}| \right)
\]

\[+ \sum_{i=1}^{n} f_i \left( \phi_i - \frac{1}{2} \sum_{j \neq i} |I_{i,j}| \right). \quad (39)
\]

using previous notations.

These expressions are useful for a deep understanding of these integrals, since all coefficients are positive: thus the contribution of each term to the value of the integral becomes clear. The following interpretation can be done:

- the Choquet integral is formed of conjunctive terms, disjunctive terms, and linear terms. A conjunctive term appears whenever the interaction index is positive, while a disjunctive one appears when the interaction index is negative. The linear part is weighted by the Shapley value minus all interactions (positive or negative). Putting the Choquet integral into the form:

\[
\mathcal{C}_\mu(f) = \sum_{i,j} \alpha_{ij} (f_i \land f_j) + \sum_{i,j} \beta_{ij} (f_i \lor f_j) + \sum_i \gamma_i f_i
\]

and remarking that due to \( \sum_i \phi_i = \mu(N) = 1 \), we have:

\[
\sum_{i,j} \alpha_{ij} + \sum_{i,j} \beta_{ij} + \sum_i \gamma_i = 1
\]

we conclude that any Choquet integral w.r.t a 2-additive measure is a convex combination of conjunctions, disjunctions and linear terms. The reciprocal also holds since any coefficient in the combination can take the value 1.

- the Šipos integral, although more complicated, is very similar to the Choquet integral, except that negative and positive numbers are processed differently. In fact, disjunctions turn to conjunctions when the sign changes. Another remarkable fact is that pairs \((f_1, f_2)\) of values with different signs are not considered.
This interpretation is of high interest in decision making.

To conclude this section, we present a graphical interpretation of the Choquet integral when \( n = 2 \) [14]. In this case, formula (38) becomes:

\[
C_\mu(f) = \begin{cases} 
(f_1 \land f_2)I_{12} + f_1(\phi_1 - \frac{1}{2}I_{12}) + f_2(\phi_2 - \frac{1}{2}I_{12}), & \text{if } I_{12} \geq 0 \\
(f_1 \lor f_2)I_{12} + f_1(\phi_1 + \frac{1}{2}I_{12}) + f_2(\phi_2 + \frac{1}{2}I_{12}), & \text{if } I_{12} \leq 0,
\end{cases}
\]

which is in fact the general expression for Choquet integral when \( n = 2 \), since in this case, at most 2-additive measures coincide with general fuzzy measures. Clearly, all possible Choquet integrals are obtained when \( \phi_1, \phi_2 \) and \( I_{12} \) vary on their domain. Using Property 6, this domain is defined by:

\[
\begin{align*}
\phi_1 - \frac{1}{2}I_{12} &\geq 0 \\
\phi_2 - \frac{1}{2}I_{12} &\geq 0 \\
\phi_1 + \frac{1}{2}I_{12} &\geq 0 \\
\phi_2 + \frac{1}{2}I_{12} &\geq 0
\end{align*}
\]

Recalling that \( \phi_1 + \phi_2 = 1 \), this domain can be represented with the \((\phi_1, I_{12})\) coordinates only, and is the shaded area on figure 1. In other words, the Choquet integral is the convex closure of the four vertices of Fig. 1. We further study these vertices, and the two axes.

Let us consider the horizontal axis, where \( I_{12} = 0 \). In the case \( n = 2 \), this is equivalent to say that the measure is additive, and thus the Choquet integral is a weighted sum\(^1\), with weights \( \mu(\{1\}) \) and \( \mu(\{2\}) \), which coincide with \( \phi_1 \) and \( \phi_2 \). Thus the horizontal axis represent the set of all possible weighted sums

Let us examine the vertical axis, where \( \phi_1 = \phi_2 = \frac{1}{2} \). Assuming \( I_{12} > 0 \) (the converse case leads to the same result), formula (40) becomes:

\[
C_\mu(a_1, a_2) = \frac{1}{2}a(1 + I_{12}) + \frac{1}{2}a(1 - I_{12})
\]

We recognize here an ordered weighted average\(^2\), with weights \( \frac{1}{2}(1 + I_{12}) \) and \( \frac{1}{2}(1 - I_{12}) \). Moreover, since any OWA operator is such that \( \phi_1 = \phi_2 = \frac{1}{2} \) when \( n = 2 \), the vertical axis is the locus of all possible OWA operators. The upper vertex \( (I_{12} = 1) \) corresponds to the minimum operator, and the lower vertex to the maximum operator, as it can be seen from (41).

---

\(^1\) A weighted sum is any expression of the form \( g(x_1, x_2) = w_1x_1 + w_2x_2 \), with \( w_1, w_2 \in \mathbb{R}^+ \), and \( w_1 + w_2 = 1 \).

\(^2\) An ordered weighted average is any expression of the form \( g(x_1, x_2) = w_1x_1 + w_2x_2 \), with \( w_1, w_2 \in \mathbb{R}^+ \), and \( w_1 + w_2 = 1 \).
6 The ordinal case: the Sugeno integral

So far we have dealt with real-valued set functions. In this section, we address the case of ordinal structures. We suppose that set functions are valued on a linearly ordered set $E$. This corresponds to applications of evaluations on an ordinal scale, where only order makes sense, and any arithmetical operation is forbidden. In this case, the Choquet and Šipoš integrals are of no use, and one has to turn to other integrals. The basic material here is the Sugeno integral [31], defined as follows, for any function $f$ valued on $[0,1]$:

$$S_\mu(f) := \bigvee_{i=1}^n [f(\delta) \wedge \mu(A(\delta))],$$

with same notations as for the Choquet integral (24). This definition uses only minimum and maximum operators, so it can be used for any function and fuzzy measures valued on a linearly ordered set. Using the residuated difference operator $\prec$ defined by:

$$a \prec b := \begin{cases} a, & \text{if } a > b \\ \emptyset, & \text{otherwise} \end{cases}$$

Figure 1: Interpretation in term of interaction index
the above expression can be rewritten as [25]:

$$S_{\mu}(f) := \bigvee_{i=1}^{n} ([f(i) \# f(i-1)] \wedge \mu(A_{i})).$$ (44)

The similarity with the Choquet integral is striking, replacing sum by maximum, and product by minimum. This shows that the Sugeno integral is a translation of the Choquet integral in the ordinal world. Other types of integrals obtained by changing operators are studied at length in [25, 18].

In the sequel, we show how constructions similar to the cardinal case can be done in this context, namely set functions transformations and the definition of symmetric and asymmetric integrals.

As we need the notion of “negative” number on ordinal scales for the definition of symmetric and asymmetric integrals, we begin by constructing a suitable scale with suitable ordinal operators. Let $E$ be a linearly ordered scale with $2k + 1$ degrees defined by:

$$e_{-k} < e_{-k+1} < \ldots < e_{-1} < e_{0} < e_{1} < \ldots < e_{k-1} < e_{k}$$

centered around $e_{0}$. We speak of “negative” value for any degree with a negative index, and denote $E^{+} := \{e_{0}, e_{1}, \ldots, e_{k}\}$ and $E^{-} := E \setminus E^{+}$. By commodity, we denote $e_{0}$ by $\emptyset$ and $e_{k}$ by $1$. We introduce for every $e_{i}$ in $E$ its symmetrical element $-e_{i}$ defined $-e_{i} := e_{-i}$ (symmetric ordinal scale). We extend on $E$ the usual operations $\lor, \land$ defined on $E^{+}$, so as to have an algebraic structure like a ring. Denoting $\oplus, \otimes$ these extended operators, called symmetric maximum and symmetric minimum respectively, it would be desirable to have properties like $-(a \oplus b) = (-a) \ominus (-b)$, $a \ominus (-b) = a \ominus b$, and $-(a \otimes b) = (-a) \otimes (-b) = a \otimes (-b)$, where $\oplus$ is the residuated difference (43) properly extended on $E$. Here, $\oplus, \otimes$ play the role of addition and product respectively.

The following definitions are suitable for our purpose [13]. The symmetric difference is defined for any $a, b$ on $E^{+}$ by:

$$a \oplus b := \begin{cases} a, & \text{if } a > b \\ \emptyset, & \text{if } a = b \\ -b, & \text{otherwise.} \end{cases}$$ (45)

Using this definition, the symmetric maximum is defined in Table 3. Then, the symmetric minimum is defined in Table 4. With these definitions, we can show the following result.

**Property 12** The structure $(E, \oplus, \otimes)$ has the following properties.

(i) $\otimes$ is commutative.

(ii) $\emptyset$ is the unique neutral element of $\otimes$, and the unique absorbant element of $\otimes$.  

15
<table>
<thead>
<tr>
<th>$a \odot b$</th>
<th>$b &lt; \emptyset$</th>
<th>$b = \emptyset$</th>
<th>$b &gt; \emptyset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a &lt; \emptyset$</td>
<td>$a \land b$</td>
<td>$a$</td>
<td>$b = \neg a$</td>
</tr>
<tr>
<td>$a = \emptyset$</td>
<td>$a$</td>
<td>$\emptyset$</td>
<td>$b$</td>
</tr>
<tr>
<td>$a &gt; \emptyset$</td>
<td>$a = \neg b$</td>
<td>$a$</td>
<td>$a \lor b$</td>
</tr>
</tbody>
</table>

Table 3: Definition of the symmetric maximum $\odot$

<table>
<thead>
<tr>
<th>$a \odot b$</th>
<th>$b &lt; \emptyset$</th>
<th>$b = \emptyset$</th>
<th>$b &gt; \emptyset$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a &lt; \emptyset$</td>
<td>$</td>
<td>a</td>
<td>\land</td>
</tr>
<tr>
<td>$a = \emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$a &gt; \emptyset$</td>
<td>$-(a \land</td>
<td>b</td>
<td>)$</td>
</tr>
</tbody>
</table>

Table 4: Definition of the symmetric minimum $\odot$

(iii) $a \odot -a = \emptyset$, for all $a \in E$.

(iv) $-(a \odot b) = (-a) \odot (-b)$.

(v) $\odot$ is associative on $E^+$ and on $E^-$.

(vi) $\odot$ is commutative.

(vii) $\mathbb{I}$ is the unique neutral element of $\odot$, and the unique absorbant element of $\odot$.

(viii) $\odot$ is associative on $E$.

(ix) $\odot$ is distributive w.r.t $\odot$ in $E^+$ and $E^-$.

The associativity of $\lor$ and distributivity do not hold on $E$ in general.

The second step is to address set functions (in fact, we restrict to fuzzy measures here) and their transformations. In this ordinal context, a fuzzy measure is defined naturally as a set function $\mu : \mathcal{P}(N) \rightarrow E^+$ such that $\mu(\emptyset) = 0$, $\mu(N) = \mathbb{I}$, and $A \subset B$ implies $\mu(A) \leq \mu(B)$. Following Berge [1], who presents the Möbius function as a powerful tool for inversion formulas over posets, we define the ordinal Möbius transform of $\mu$, denoted $m^\mu_\odot$ as the solution of the equation:

$$\mu(A) = \bigodot_{B \subset A} m^\mu_\odot(B).$$

(46)

In fact, unlike the cardinal case, there is not a single solution to this equation. Previous studies by the author [9, 11] and related works of Mesiar [23] and Marichal [22] have shown that the smallest solution is given by:

$$m^\mu_\odot(A) := \begin{cases} \mu(A), & \text{if } \mu(A) > \mu(A \setminus i), \forall i \in A \\ \emptyset, & \text{otherwise}, \end{cases}$$

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and the greatest one is simply \( m_\mu \equiv \mu \). Any set function comprised between these two solutions is a solution. We take as definition of the Möbius transform the lower bound, which is a non-negative function, so that we can use from now on usual \( \lor \) and \( \land \) operators. If there is no fear of ambiguity, the superscript \( \mu \) can be dropped.

Interestingly enough, the Möbius transform can be expressed as:

\[
m_\mu(A) := \bigvee_{B \subset A, \lvert A \setminus B \rvert \text{ even}} \mu(B) \lor \bigvee_{B \subset A, \lvert A \setminus B \rvert \text{ odd}} \mu(B)
\]

which is an exact transcription of the cardinal case (see (7)). However, if \( \mu \) is not monotonic, the above formula is no more solution of (46) in general.

The ordinal Möbius transform has many interesting properties. E.g., let us consider any possibility measure \( \Pi \), i.e. a fuzzy measure such that \( \Pi(A \cup B) = \Pi(A) \lor \Pi(B) \) for all \( A, B \subseteq N \). Then its Möbius transform \( m_\Pi \) is:

\[
m_\Pi(\{i\}) = \Pi(\{i\}), \quad \forall i \in N, \quad \text{and} \quad m_\Pi(A) = 0, \quad \forall A \subseteq N, \lvert A \rvert > 1.
\]

Thus we have the same kind of result as with probability measures in the cardinal context.

Attempts have been done by Grabisch to define an interaction transform in this ordinal context [11]. Changing in a suitable way operators in formula (9), and taking into account some properties which are counterparts of the cardinal case (see below), the following equation defines the ordinal interaction transform of a fuzzy measure \( \mu \):

\[
I_\mu^\Pi(A) := \bigvee_{B \subset N \setminus A} \bigvee_{\substack{C \subseteq A \atop \lvert A \setminus C \rvert \text{ even}}} \mu(B \cup C) \lor \bigvee_{\substack{C \subseteq A \atop \lvert A \setminus C \rvert \text{ odd}}} \mu(B \cup C).
\]

(47)

For \( A = \{i\} \), we get the ordinal Shapley value:

\[
\phi_\mu(i) := \bigvee_{A \subseteq N \setminus i} \lceil \mu(A \cup i) - \mu(A) \rceil.
\]

(48)

The ordinal Shapley value fulfills the following properties:

(A1) \( \bigvee_{i \in N} \phi_\mu(i) = \mu(N) \) (sharing of \( \mu(N) \)).

(A2) if \( i \) is such that \( \mu(A \cup i) = \mu(A) \) for every \( A \subseteq N \setminus i \), then \( \phi_\mu(i) = 0 \) (null player).

(A3) if \( i, j \) are such that \( \mu(A \cup i) = \mu(A \cup j) \) for every \( A \subseteq N \setminus \{i, j\} \), then \( \phi_\mu(i) = \phi_\mu(j) \) (symmetric players).

These properties are counterparts of the properties of the (original) Shapley value [29]. However, contrary to what is wrongly claimed in [11], the “maxitivity” does not hold: in general \( \phi_\mu^\lor \neq \phi_\mu \lor \phi_\nu \), where \( \mu, \nu \) are two fuzzy
measures. Lastly, the ordinal interaction transform can be expressed through the ordinal Möbius transform by:

$$I^o_p(A) = \bigvee_{B \subseteq N \setminus A} m^o_p(A \cup B).$$  \hspace{1cm} (49)

Again, the similarity with the cardinal case has to be noted.

The last step is to address the definition of the Sugeno integral over $E$, the expression over $E^+$ being given by equation (42), where $f$ is any function from $N$ to $E^+$, and $\mu$ is a fuzzy measure in the sense defined above. As in the cardinal case, it appears that the Sugeno integral can be viewed as an extension of fuzzy measures (or pseudo-Boolean functions). Specifically, any fuzzy measure can be expressed on the basis of unanimity games, by:

$$\mu(A) = \bigvee_{B \subseteq N} (m^o_p(B) \land u_B(A))$$  \hspace{1cm} (50)

which is the counterpart of (3) and (21) in the ordinal case. The extension of this formula on $(E^+)^n$ gives rise to the Sugeno integral:

$$S^o_\mu(f) = \bigvee_{A \subseteq N} \left[ m^o_\nu(A) \land \left( \bigwedge_{i \in A} f_i \right) \right].$$  \hspace{1cm} (51)

Remark the analogy with the Choquet integral (see (25)).

Based on formulas (44) and (28), the following expression defines the symmetric Sugeno integral [13]:

$$\tilde{S}^o_\mu(f) := \left[ \bigotimes_{i=1}^p \left( f_i \otimes \mu((i_1, \ldots, i)) \right) \right] \otimes \left[ \bigotimes_{i=p+1}^n \left( f_i \otimes \mu((i_1, \ldots, n)) \right) \right],$$  \hspace{1cm} (52)

for any $f \in E^n$, with $f_{i(1)} \leq \cdots \leq f_{i(p)} < 0 \leq f_{i(p+1)} \leq \cdots \leq f_{i(n)}$. Note that the above expression is well-defined, i.e. there is no problem of associativity of $\otimes$, since $\bigotimes_{i=1}^p$ operates only on $E^-$, and $\bigotimes_{i=p+1}^n$ only on $E^+$. The symmetric Sugeno integral indeed satisfies the property of symmetry:

$$\tilde{S}^o_\mu(-f) = -\tilde{S}^o_\mu(f)$$  \hspace{1cm} (53)

for any fuzzy measure $\mu$ and any $f \in E^n$.

Also, $\tilde{S}$ can be expressed in terms of the ordinal Möbius transform, in a way which is the counterpart of (32):

$$\tilde{S}^o_\mu(f) = \bigotimes_{A \subseteq N} \left( m^o_\nu(A) \otimes \left[ \bigwedge_{i \in A} f_i^{+} \right] \otimes \left[ \bigwedge_{i \in A} f_i^{-} \right] \right).$$  \hspace{1cm} (54)
where $f^+_i := f_i \land \emptyset$, and $f^-_i := (-f_i) \land \emptyset$. This shows the adequacy of the whole construction.

The case of the asymmetric integral is not necessary to consider, if it can be thought, as in the cardinal case, that the expression of the asymmetric integral does not depend on the position of the zero of the scale. Then, the zero can be put at the lower extremity, and we are working on a scale with only positive values, where the usual definition of the Sugeno integral works. Another possibility is to take as definition:

$$\tilde{S}_\mu(a) = S_\mu(a^+) \sim S_\mu(a^-)$$

where $\tilde{S}_\mu$ stands for the asymmetric integral. But this needs a proper definition of $\tilde{\mu}$, which requires more structure on $E$ than what we have defined so far.

**References**


