A new investigation about the core and Weber set of multichoice games

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Multichoice games have been introduced by Hsiao and Raghavan as a generalization of classical cooperative games. An important notion in cooperative game theory is the core of the game, as it contains the rational imputations for players. We propose two definitions for the core of a multichoice game, the first one is called the precore and is a direct generalization of the classical definition. We show that the precore coincides with the definition proposed by Faigle, and that it contains unbounded imputations, which makes its application questionable. A second definition is proposed, imposing normalization at each level, causing the core to be a convex closed set. We study its properties, introducing balancedness and marginal worth vectors, and defining the Weber set and the pre-Weber set. We show that the classical properties of inclusion of the (pre)core into the (pre)-Weber set as well as their equality remain valid. A last section makes a comparison with the core defined by Tijs et al.

Key words: multichoice game ; lattice ; core
MSC2000 Subject Classification: Primary: 91A12, ; Secondary: ,
OR/MS subject classification: Primary: cooperative game, ; Secondary: ,

1. Introduction

In cooperative game theory, one of the central problems is to define an imputation, i.e., a way of sharing among players the total worth of a game, if they all join the grand coalition. There are two basic ways to do this. The first one is to define rational axioms such a sharing should satisfy, e.g., the null or dummy player axioms, symmetry, linearity, etc., with the hope that a unique sharing satisfies the set of axioms. This leads to several definitions of values or solution concepts, the most famous ones being the Shapley value [13] and the Banzhaf value [1]. The other way is to find a sharing so that no subcoalition has interest to form, that is, the sum of imputations for a given subcoalition is always greater or equal to the worth of this subcoalition. The set of such imputations, whenever they exist, is called the core of the game. Classical results show under which conditions the core is non empty, and give the structure of the core when the game is convex. A related notion is the Weber set (see, e.g., [4]), which is proven to always contain the core, with equality attained in case of convexity.

Many generalizations of the classical notion of cooperative game in characteristic form have appeared the last decade, in order to model in a more accurate way real situations. We may cite games with restricted cooperation of Faigle [7, 8], where admissible coalitions should verify some precedence constraints, multichoice games of Hsiao and Raghavan [12], where each player is allowed to choose among a totally ordered set of actions, fuzzy games [3] which can be considered as a continuous extension of multichoice games, bi-cooperative games of Bilbao [2], where two coalitions are considered, one being opponent of the other, etc. All these examples can be thought as particular instances of games defined on a distributive lattice structure, see a general exposition of this in [10, 11], and also Faigle and Kern [8].

A natural question is then to try to define the core of a game on a lattice, or of some of the most useful examples cited above. In this paper, we focus on multichoice games (Section 3), and propose a definition of the core and the Weber set. As it will be shown in Section 4 the situation appears to be more complex than for the classical case, although similar results still hold. A first immediate generalization of the classical definition leads to what we call the precore, which happens to be a convex polyhedron with infinite directions. We propose to call core a particular closed convex subset of it, satisfying some normalization constraint. Similarly to the classical case, we call pre-Weber set the convex hull of the additive games induced by marginal worth vectors, and the Weber set is a particular subset of it. We show that in case of convexity, we still have equality between the core and the Weber set, and between the closed convex part of the precore and the pre-Weber set. Moreover, the inclusion of the core into the Weber set holds in any case, as well as the inclusion of the convex closed part of the precore into the pre-Weber set.
We compare our results with previous works of Faigle and Tijs et al. in Section 5. We find that our precore is the core defined by Faigle \cite{Faigle}. The relation with the core of Tijs \cite{Tijs} appears to be less simple, since it is a subset of our precore. Our results show that the set of vertices of the Weber set found by Tijs is unnecessarily large.

The interest of the definition of the core we propose is that it keeps the original meaning of the set of imputations such that no subcoalition has interest to form, subcoalition being here replaced by the more general notion of participation profile with unequal levels of participation.

We begin by setting our notations and recall classical results in Section 2. Then Section 3 presents the basic material for games on lattices and in particular multichoice games. In Section 4, we give our definitions of the core and the Weber set, and study their properties. Lastly, Section 5 compares our approach with previous work.\footnote{A preliminary and short version of this paper has been presented at 4th Logic, Game Theory and Social Choice meeting, Caen, France, June 2005.}

2. A review of classical results (see, e.g., Driessen \cite{Driessen}) Throughout this paper, we consider a finite set of players $N := \{1, \ldots, n\}$. A \textit{coalition} is any subset of $N$, representing the set of players who effectively participates to the game. The \textit{grand coalition} is $N$ itself. A \textit{transferable utility game in characteristic form} (called hereafter \textit{game} for simplicity) is any function $v : 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. $v(A)$ is the asset or income the coalition $A$ will win if all players in $A$ equally participate to the game.

A game is said to be \textit{convex} if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. A game is said to be \textit{additive} if $v(S \cup T) = v(S) + v(T)$, for all disjoint $S, T \subseteq N$. If $v$ is additive, it suffices to know only $v(\{i\})$, $i \in N$. The set of additive games is denoted $\mathcal{A}(2^N)$.

It is useful for the sequel to reconsider the above definitions in terms of lattices. A game is a real-valued function defined on the Boolean lattice $(2^N, \subseteq)$, vanishing at the bottom ($\emptyset$) of the lattice. Union and intersection are the sup $\lor$ and inf $\land$ operations of the lattice, so that a general form of convexity writes $v(x \lor y) + v(x \land y) \geq v(x) + v(y)$. This is usually called \textit{supermodularity}.

For any lattice $(L, \leq)$, a \textit{chain} is a totally ordered sequence of elements of $L$, e.g., $x < y < z < \cdots$. A chain is maximal if no superset of it is again a chain. In a finite distributive lattice, maximal chains going from bottom $\bot$ to top $\top$ have all the same length, and are of the form $\bot < x < y < z < \cdots < \top$, where $x < y$ means that $x$ is covered by $y$.

In the Boolean lattice $(2^N, \subseteq)$, a maximal chain between $\emptyset$ and $N$ is of the form $\emptyset \subset \{i\} \subset \{i, j\} \subset \cdots \subset N$, and hence is associated in a bijective way with a permutation $\pi$ on $N$, defining the order in which elements $i, j, \ldots$ appear. We adopt the following notation: for any permutation $\pi$ on $N$, we define the maximal chain

$$A_0^\pi := \emptyset \subset A_1^\pi := \{\pi(1)\} \subset A_2^\pi := \{\pi(1), \pi(2)\} \subset \cdots \subset A_n^\pi := N$$

with $A_i^\pi := \{\pi(1), \ldots, \pi(i)\}$. The \textit{marginal worth vector} $x^\pi(v)$ is defined by:

$$x_{\pi(i)}^\pi(v) := v(A_i^\pi) - v(A_{i-1}^\pi), \quad i = 1, \ldots, n.$$ 

The set of marginal worth vectors is denoted $\mathcal{M}(v)$. The $i$th coordinate represents the marginal contribution of player $i$ in the chain.

Let us remark that any marginal worth vector $x^\pi$ defines an additive game $\phi^\pi$ by

$$\phi^\pi((i)) = x_i^\pi, \quad \forall i \in N,$$

with the property that $\phi^\pi(A_i^\pi) = v(A_i^\pi)$, $i = 0, \ldots, n$.

The \textit{Weber set} of $v$ is the convex hull of the set of chain vectors

$$\mathcal{W}(v) := \text{co}(\mathcal{M}(v)).$$

For any convex set $C$, we denote by $\text{Ext}(C)$ the set of its extreme points (vertices). Hence, the above equation can be written $\text{Ext}(\mathcal{W}(v)) = \mathcal{M}(v)$. 
A collection $\mathcal{B}$ of non-empty subsets of $N$ is balanced if it exists positive coefficients $\mu(S), S \in \mathcal{B}$, such that
\[ \sum_{S \ni i} \mu(S) = 1, \ \forall i \in N. \]
Any partition $\{P_1, \ldots, P_k\}$ of $N$ is a balanced collection, with coefficients $\mu(P_i) = 1, \ \forall i$. A game $v$ is balanced if for every balanced collection $\mathcal{B}$ with coefficients $\mu(S), S \in \mathcal{B}$, it holds
\[ \sum_{S \in \mathcal{B}} \mu(S)v(S) \leq v(N). \]

The core of $v$ is a set of imputations such that no subcoalition has interest to form:
\[ \mathcal{C}(v) := \{ \phi \in \mathbb{R}^n \mid \phi(N) = v(N) \text{ and } \phi(A) \geq v(A), \forall A \subseteq N \} \]
with $\phi(A) := \sum_{i \in A} \phi(i)$. Hence, it is the set of additive games dominating $v$ and coinciding on $N$. Whenever nonempty, the core is a convex set. It is reduced to the singleton $\{v\}$ if the game is additive.

The following proposition summarizes well-known results.

**Proposition 2.1** Let $v$ be a game on $N$. The following holds.

- (i) $\mathcal{C}(v) \subseteq \mathcal{W}(v)$.
- (ii) $\mathcal{C}(v) \neq \emptyset$ iff $v$ is balanced.
- (iii) $v$ is convex iff $\mathcal{C}(v) = \mathcal{W}(v)$ (i.e., the set of chain vectors is the set of vertices of the core).

### 3. Games on lattices and multichoice games

We give a brief introduction to games defined on lattices (see [10][11] for a more detailed presentation). We consider a set of players $N := \{1, \ldots, n\}$. Let $L_1, \ldots, L_n$ be finite distributive lattices representing the partially ordered set of actions each player can perform, and consider their product $L := L_1 \times \cdots \times L_n$, together with the product order, i.e., for $x := (x_1, \ldots, x_n), y := (y_1, \ldots, y_n) \in L$, $x \leq y$ if $x_i \leq y_i$ for $i = 1, \ldots, n$. Supremum and infimum are also defined coordinatewise. The top and bottom of $L$ are denoted $\top, \bot$.

A game on $L$ is any function $v : L \to \mathbb{R}$ such that $v(\bot) = 0$. We denote by $G(L)$ the set of games on $L$. Specifically, $L_i$ is the ordered set of possible actions player $i$ has at his disposal. The bottom element of $L_i$ indicates no participation to the game by player $i$, while the top element indicates full participation. A given $x \in L$ is then a profile of participation of all players, indicating for each player the action chosen. This replaces the notion of coalition.

We give some examples, to recover known concepts. Classical cooperative games correspond to $L_i = \{0, 1\}$ for all $i \in N$, “0” indicating no participation and “1” full participation; hence $L$ is the Boolean lattice $2^N$. Multichoice games of Hsiao and Raghavan correspond to $L_i = \{0, 1, \ldots, l_i\}$ for all $i \in N$, where $1, 2, \ldots, l_i$ denotes the ordered participation levels of a player. Bi-cooperative games may correspond to the case $L_i = \{0, 1, 2\}, i \in N$, where “0” corresponds to participation against, “2” to participation in favor, and “1” no participation, although a better notation would be $L_i = \{-1, 0, 1\}$. In fact, bi-cooperative games, as well as ternary voting games of Felsenthal and Machover [3], correspond to the simplest case of bipolar games, as defined by Grabisch in [10], and are not isomorphic to multichoice games with $l = 2$ (this point is however out of the scope of this paper).

In this paper, we focus on multichoice games; however our definition is slightly more general than the original one.

**Definition 3.1** Let $L_i := \{0, 1, 2, \ldots, l_i\}, i \in N$, and consider the product lattice $L = L_1 \times \cdots \times L_n$. A multichoice game is any function $v : L \to \mathbb{R}$ such that $v(0, 0, \ldots, 0) = 0$. A $l$-choice game is a multichoice game where $l_1 = l_2 = \cdots = l_n = l$.

Note that $(0, 0, \ldots, 0)$ is the bottom element $\bot$ of $L$, and “0” denotes non participation. The top element of $L$ is denoted $\top := (l_1, \ldots, l_n)$ as usual. Any $x := (x_1, \ldots, x_n) \in L$ is a participation profile, indicating
the participation level of each player. We recall that (see above):

\[ x \leq y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2 \text{ and } \cdots \text{ and } x_n \leq y_n \]

\[ x \vee y := (x_1 \vee y_1, \ldots, x_n \vee y_n) \]

\[ x \wedge y := (x_1 \wedge y_1, \ldots, x_n \wedge y_n). \]

For ease of notation, we denote by \((x_i, k_i)\) the participation profile where player \(i\) plays at level \(k \in L_i\), and other players play at levels defined by the participation profile \(x \in L\). We put for convenience \(x_{-i} = \prod_{j \neq i} L_j\). A useful particular case is the situation where the \(i\)-th player plays at the \(k\)-th action level, \(k > 0\), the others doing nothing, which is denoted by \((0_{-i}, k_i)\), or \(k_i\) for short. Accordingly, we introduce \(\tilde{L} := \{k_i \mid k \in \{1, \ldots, l_i\}, i \in N\} \subseteq L\).

We denote by \(C_m(L)\) the set of maximal chains of the lattice \(L\). The number of maximal chains is known to be (see, e.g., Faigle and Kern [3])

\[
|C_m(L)| = \frac{(l_1 + l_2 + \ldots + l_n)!}{l_1!l_2!\ldots l_n!}.
\]

The definition of convexity and additivity follows the one of functions defined on any lattice.

**Definition 3.2** Let \(v\) be a multichoice game on \(L\). Then

(i) \(v\) is convex if \(v(x \vee y) + v(x \wedge y) \geq v(x) + v(y)\), for all \(x, y \in L\).

(ii) \(v\) is additive if for every \(x, y \in L\) such that \(x \wedge y = \bot\), it holds \(v(x \vee y) = v(x) + v(y)\).

Let us give an equivalent definition for additive games. We define mappings \(v_i : L_i \rightarrow \mathbb{R}\) by \(v_i(k) := v(0_{-i}, k_i), k \in L_i\). Assuming additivity of \(v\), we have \(v(0_{-(i,j)}, k_i, k_j') = v_i(k) + v_j(k')\). Hence, by repeating this process, we finally get for any \(x \in L\)

\[
v(x) = \sum_{i=1}^{n} v_i(x_i).
\]

Reciprocally, if \(v\) can be written under the form (2), then it is additive since taking \(x, y \in L\) such that \(x \wedge y = 0\) implies that for each \(i \in N\), either \(x_i = 0_i\) or \(y_i = 0_i\). Hence \(v(x \vee y) = v(x) + v(y)\) holds, and we have shown:

**Proposition 3.1** A game \(v\) is additive iff for all \(x \in L\), Eq. (2) holds, with \(v_i : L_i \rightarrow \mathbb{R}\), and \(v_i(k) = v(0_{-i}, k_i)\), for all \(i \in N\) and \(k \in L_i\).

We denote by \(A(L)\) the set of additive games on \(L\), and will often denote additive games by \(\phi\) instead of \(v\).

Remark that we can write \(\phi(x) = \sum_{i=1}^{n} \phi(\tilde{x}_i)\) for additive games, where we recall that \(\tilde{x}_i\) stands for \((0_{-i}, x_i)\). This shows that an additive game can be identified bijectively with a vector in \(\mathbb{R}^{\sum l_i}\) or \(\mathbb{R}^{\tilde{L}}\). Hence, it is convenient to number the coordinates by the corresponding elements of \(\tilde{L}\). For example, with 2 players and \(l_1 = 2\) and \(l_2 = 3\), this would give

\[
\phi = (\phi_{1_1}, \phi_{2_1}, \phi_{1_2}, \phi_{2_2}, \phi_{3_2}),
\]

instead of the uninformative \((\phi_1, \ldots, \phi_3)\). With this notation, we can write:

\[
\phi(x) = \sum_{i=1}^{n} \phi(\tilde{x}_i).
\]

**Remark 3.1** The concepts introduced in Def. [3] remain valid for games on any lattice \(L = \prod_{i=1}^{n} L_i\). However, the equivalence between the two definitions of additivity is lost in general. In this last case, the definition in Def. [3] should be kept.
4. The core and Weber set of multichoice games

The following definition is a direct transposition of the classical definition.

**Definition 4.1** The precore of a multichoice game \( v \) on \( L \) is defined by
\[
\text{PC}(v) := \{ \phi \in A(L) \mid \phi(x) \geq v(x), \forall x \in L, \ \text{and} \ \phi(\top) = v(\top) \}.
\]

Note that this defines an imputation for every player and every level of participation. Clearly, the precore is a convex set. Let us consider the following example as an illustration.

**Example 4.1** We consider a 2-choice game with two players, hence \( L := \{0, 1, 2\}^2 \). The conditions on \( \phi \) to be element of the core write:
\[
\begin{align*}
\phi(2, 0) + \phi(0, 2) &= v(2) \\
\phi(1, 0) &\geq v(1, 0) \\
\phi(0, 1) &\geq v(0, 1) \\
\phi(1, 0) + \phi(0, 1) &\geq v(1, 1).
\end{align*}
\]

Remark that \( \phi(1, 0) \) and \( \phi(0, 1) \) may be taken arbitrarily large. This shows that in general, \( \text{PC}(v) \) is an unbounded convex polyhedron, hence it can be written as the sum of a polytope \( \text{PC}^F(v) := \text{co}(\text{Ext}(\text{PC})) \) and infinite directions.

On the point of view of interpretation, this is not acceptable since imputation may become infinite. This drawback could be avoided if normalization occurs not only on the last level, i.e., \( \phi(\top) = v(\top) \), but also on each level 1, 2, ..., \( l \). This motivates the next definition.

**Definition 4.2** The core of a multichoice game \( v \) on \( N \) is defined as:
\[
\mathcal{C}(v) := \{ \phi \in A(L) \mid \phi(x) \geq v(x), \forall x \in L, \ \text{and} \ \phi(k \land l_1, \ldots, k \land l_n) = v(k \land l_1, \ldots, k \land l_n), k = 1, \ldots, \text{max } l_j \}.
\]

Note that the pre-core always contains the core. For \( l \)-choice games, the normalization condition simplifies into \( \phi(k, k, \ldots, k) = v(k, k, \ldots, k) \), \( k = 1, \ldots, l \).

Obviously the core is a convex set, where all variables are bounded, hence it is a convex polytope.

Our definition of the core permits to keep the classical interpretation of the core as an imputation vector, such that no participation profile with unequal (unless maximal) level has interest to form. The difference with the classical case is that we may have a participation profile which is not maximum everywhere, but with equal level for all players, such as \((1, 1, \ldots, 1), (2, 2, \ldots, 2), \ldots\).

We introduce now balancedness for the core and precore. A collection \( \mathcal{B} \) of elements of \( L \setminus \{\bot\} \) is *pre-balanced* if it exists positive coefficients \( \mu(x), x \in \mathcal{B} \), such that
\[
\sum_{x \in \mathcal{B} | x_i = 1} \mu(x) = 1, \ \text{for all } i \in N,
\]
\[
\sum_{x \in \mathcal{B} | x_i = 0} \mu(x) = 0, \ \text{for all } k = 1, \ldots, l_i - 1, \ i \in N.
\]

**Definition 4.3** A multi-choice game \( v \) is pre-balanced if for every pre-balanced collection \( \mathcal{B} \) of elements of \( L \setminus \{0\} \) with coefficients \( \mu(x), x \in \mathcal{B} \), it holds
\[
\sum_{x \in \mathcal{B}} \mu(x)v(x) \leq v(\top).
\]

**Proposition 4.1** A multi-choice game has a non-empty precore if and if only it is pre-balanced.
\textbf{Proof.} Nonemptiness of the precore is equivalent to find \(\phi \in A(L)\) such that
\[
\sum_{i \in N} \phi(0_{-i}, x_i) \geq v(x), \forall x \in L, \text{ and } \phi(\top) = \sum_{i \in N} \phi(0_{-i}, l_i) = v(\top),
\]
which in turn is equivalent to the fact that the following linear program
\[
\min z = \sum_{i \in N} \phi(0_{-i}, l_i) \text{ subject to } \sum_{i \in N} \phi(0_{-i}, x_i) \geq v(x), \forall x \in L
\]
has an optimal solution with value \(z^* \leq v(\top)\). Indeed, any optimal solution is such that \(\sum_{i \in N} \phi(0_{-i}, l_i) = v(\top)\), and so lies in the core. Conversely, any \(\phi\) in the core satisfies the contraints, and is such that \(\phi(\top) = v(\top)\). Hence \(z^* \leq v(\top)\) is true.

Its dual problem writes
\[
\max q = \sum_{x \in L} \mu(x)v(x) \text{ subject to } \\
\mu(x) \geq 0, \quad x \in L \\
\sum_{x \in L_{-i}} \mu(x_{-i}, l_i) = 1, \quad i \in N \\
\sum_{x \in L_{-i}} \mu(x_{-i}, k_i) = 0, \quad 1 \leq k \leq l_i - 1, \quad i \in N.
\]
By the duality theorem, \(z^* = q^* \leq v(\top)\). Hence, nonemptiness of the core is equivalent to the existence of an optimal solution \(\mu^*\) such that \(\sum_{x \in L} \mu^*(x)v(x) \leq v(\top)\) and satisfying the constraints, which means exactly that \(v\) is pre-balanced. \(\square\)

A collection \(B\) of elements of \(L \setminus \{\bot\}\) is \textit{balanced} if it exists positive coefficients \(\mu(x), x \in B\), such that
\[
\sum_{x \in B \mid x_i = l_i} \mu(x) = (\max_j l_j) - l_i + 1, \quad \text{for all } i \in N, \\
\sum_{x \in B \mid x_i = k} \mu(x) = 1, \quad \text{for all } k = 1, \ldots, l_i - 1, \quad i \in N.
\]
Remark that for \(l\)-choice games, the above conditions simplify into \(\sum_{x \in B \mid x_i = k} \mu(x) = 1, \text{for all } k = 1, \ldots, l\) and \(i \in N\).

\textbf{Definition 4.4} A \textit{multichoice game} \(v\) is balanced if for every balanced collection \(B\) of elements of \(L \setminus \{\bot\}\) with coefficients \(\mu(x), x \in B\), it holds
\[
\sum_{x \in B} \mu(x)v(x) \leq \sum_{k=1}^{\max_j l_j} v(k \land l_1, \ldots, k \land l_n)
\]
\textbf{Proposition 4.2} A \textit{multichoice game} has a nonempty core if and only if it is balanced.

\textbf{Proof.} The proof is similar to the one of Prop. \textbf{4.1} Nonemptiness of the core is equivalent to find \(\phi \in A(L)\) such that
\[
\sum_{i \in N} \phi(0_{-i}, x_i) \geq v(x), \forall x \in L, \text{ and } \phi(k \land l_1, \ldots, k \land l_n) = \\
\sum_{i \in N} \phi(0_{-i}, k \land l_i) = v(k \land l_1, \ldots, k \land l_n), \quad \forall k = 1, \ldots, \max_j l_j,
\]
which is equivalent to the fact that the following linear program:
\[
\min z = \sum_{i \in N} \left[ \sum_{1 \leq k < l_i} \phi(0_{-i}, k_i) + ((\max_j l_j) - l_i + 1)\phi(0_{-i}, l_i) \right] \\
\text{subject to } \sum_{i \in N} \phi(0_{-i}, x_i) \geq v(x), \forall x \in L.
\]
has an optimal solution $z^* \leq \sum_{k=1}^{\max_j l_j} v(k \land l_1, \ldots, k \land l_n)$. Its dual problem writes

$$\max_{\mu} q = \sum_{x \in L} \mu(x)v(x) \text{ subject to}$$

$$\mu(x) \geq 0, \quad \forall x \in L$$

$$\sum_{x \in L, i} \mu(x_i, l_i) = (\max_j l_j) - l_i + 1, \quad i \in N$$

$$\sum_{x \in L, i} \mu(x_i, k_i) = 1, \quad k = 1, \ldots, l_i - 1, \forall i \in N.$$ 

The existence of an optimal solution is equivalent to the fact that $v$ is balanced.\[\qed\]

Consider a maximal chain $C := \{x^0 := \perp < x^1 < \cdots < x^{\sum_j l_j} := \top\}$ in $L$. Remark that between $x^{i-1}$ and $x^i$, only one coordinate has changed, and has been increased by exactly 1. Hence, to each maximal chain one can associate the mapping $\sigma : \{1, \ldots, \sum_j l_j\} \rightarrow L$, defined by $\sigma(i) = \vec{k}_i$, which means that at the $i$th element of the chain, the level of player $j$’s marginal worth vector has been raised from $k - 1$ to $k$, in symbols:

$$x^i = x^{i-1} \lor \vec{k}_j.$$

Consequently, one can write $x^i = \bigvee_{p=1}^i \sigma(p)$, for any element of the chain. Note that all mappings $\sigma$ (which are in fact permutations on $\{1, \ldots, \sum_j l_j\}$) may not correspond to a maximal chain. For this, $\sigma$ must fulfill: $\vec{k}_j' > \vec{k}_j$ implies $i' > i$, where $\sigma(i) := \vec{k}_j$ and $\sigma(i') := \vec{k}_j'$ (compatible mapping $\sigma$).

We define the marginal worth vector $\psi^C \in \mathbb{R}^L$ associated to $C$ and $v$ as:

$$\psi^C_{\vec{k}_j} := v(x^{\sigma^{-1}(\vec{k}_j)}) - v(x^{\sigma^{-1}(\vec{k}_j) - 1}), \quad \forall \vec{k}_j \in L.$$

From this, we define the additive game $\phi^C$ by its coordinates in $\mathbb{R}^L$:

$$\phi^C_{\vec{k}_j} := \sum_{p=1}^k \psi^C_{\vec{p}_j},$$

where $\vec{p}_j$ is the $j$th coordinate of $\vec{p}$.

The set of additive games $\phi^C$ for all maximal chains is denoted $\mathcal{P}M(v)$.

A fundamental property of $\phi^C$ is that it coincides with $v$ on $C$.

**Lemma 4.1** Let $C$ be a maximal chain on $L$. For every $x \in C$, $\phi^C(x) = v(x)$.

**Proof.** Consider a maximal chain $C := \{x^0 := \perp < x^1 < \cdots < x^{\sum_j l_j} := \top\}$ in $L$, and its associated mapping $\sigma$. Assume $\sigma(i) := \vec{k}_j$. We have:

$$\psi^C_{\vec{k}_j} = \sum_{p=1}^k \psi^C_{\vec{p}_j} = \sum_{p=1}^k \left[v(x^{\sigma^{-1}(\vec{p}_j)}) - v(x^{\sigma^{-1}(\vec{p}_j) - 1})\right] = v(x^{\sigma^{-1}(\vec{k}_j)}) = v(x^i).$$

**Definition 4.5** The pre-Weber set $\mathcal{PW}(v)$ of $v$ is defined as the convex hull of all additive games in $\mathcal{P}M(v)$:

$$\mathcal{PW}(v) \equiv \text{co}(\mathcal{P}M(v)).$$

**Theorem 4.1** If a multichoice game $v$ is convex, then any additive game in $\mathcal{P}M(v)$ is a vertex of the precore:

$$\mathcal{P}M(v) = \text{Ext}(\mathcal{PW}(v)) \subseteq \text{Ext}(\mathcal{PC}(v)).$$

**Proof.** Let $C := \{x^0 := \perp < x^1 < \cdots < x^{\sum_j l_j} := \top\}$ be a maximal chain in $L$, and consider $\phi^C$.

We first show that $\phi^C \in \mathcal{PC}(v)$. We take any $x \in L$ and show that $\phi^C(x) \geq v(x)$. By Lemma 4.1, the property is true for any $x \in C$, so let us suppose $x \notin C$. Then there exists a unique $i$ such that $x^{i+1} > x$ and $x^i \not= x$. Moreover, there exists a unique $j \in N$ such that $x^{i+1} = x_j$ and $x^i = x_j - 1$, where $x_j$ is the $j$th coordinate of $x$. In summary, we have the following situation:
By convexity of $v$, we have

$$v(x^{i+1}) - v(x^i) \geq v(x) - v(x^i \land x).$$

Note that if $x^i \land x \in C$, we are done since $\phi^C \equiv v$ on $C$, so that the above inequality writes $\phi^C(x^{i+1}) - \phi^C(x^i \land x) \geq v(x) - \phi^C(x^i \land x)$. By additivity of $\phi^C$, we have $\phi^C(x) = \phi^C(x^{i+1}) - \phi^C(x^i) + \phi^C(x^i \land x)$, which leads to $\phi^C(x) \geq v(x)$.

If $x^i \land x \notin C$, we construct a decreasing sequence $x', x'', \ldots$ which will meet the chain $C$. Specifically, let us consider $x' := x^i \land x$, and $i'$ such that $x^{i+1} > x'$ and $x' \not< x'$. The same inequality as (3) holds with $x', i'$ replacing $x, i$. If $x^i \land x' \in C$, the result can be proven as above. Indeed, we get:

$$v(x^{i+1}) - v(x^i) \geq v(x) - v(x^i \land x)$$

$$v(x'^{i+1}) - v(x') \geq v(x^i \land x) - v(x^{i'} \land x^i \land x).$$

Summing the inequalities leads to

$$v(x^{i+1}) - v(x^i) + v(x'^{i+1}) - v(x') \geq v(x) - v(x^{i'} \land x^i \land x)$$

which is equivalent to

$$\phi^C(x^{i+1}) - \phi^C(x^i) + \phi^C(x'^{i+1}) - \phi^C(x') \geq v(x) - \phi^C(x^{i'} \land x^i \land x).$$

By additivity:

$$\phi^C(x^i) = \phi^C(x^{i+1}) + \phi^C(x^i \land x) - \phi^C(x^i)$$

$$\phi^C(x) = \phi^C(x^{i+1}) + \phi^C(x') - \phi^C(x^i)$$

so that

$$\phi^C(x) = \phi^C(x^{i+1}) + \phi^C(x'^{i+1}) + \phi^C(x'^i \land x^i \land x) - \phi^C(x^i') - \phi^C(x^i).$$

Replacing into (4) leads to $\phi^C(x) \geq v(x)$.

If $x^i \land x' \notin C$, we consider $x'' := x' \land x'$ and $i''$ accordingly. Clearly the sequence $x, x', x'', \ldots$ is strictly decreasing and will meet the chain $C$ due to finiteness of $L$, latest at bottom of $L$ since $\bot \in C$. Repeating the above process proves that $\phi^C(x) \geq v(x)$, $\forall x \in L$.

It remains to show that $\phi^C$ is a vertex of the precore. Suppose there exist additive games $\phi_1, \phi_2 \neq \phi^C \in PC(v)$, and $\lambda \in (0, 1)$ such that $\phi^C = \lambda \phi_1 + (1 - \lambda) \phi_2$. Because we have $\phi^C(x^i) = v(x^i)$ for any $x^i \in C$, we have $v(x^i) = \lambda \phi_1(x^i) + (1 - \lambda) \phi_2(x^i)$. But $\phi_1(x^i) \geq v(x^i)$ for all $x^i \in C$, $k = 1, 2$, hence necessarily $\phi_1(x^i) = \phi_2(x^i) = v(x^i)$, i.e., $\phi_1 = \phi_2 = \phi^C$, a contradiction.

**Definition 4.6** A restricted maximal chain $C_r = \{x^0 := \bot < x^1 < \cdots < x_{\Sigma_{i=1}^n} := \top\}$ in $L$ is a maximal chain passing by all $(k \land l_1, \ldots, k \land l_n)$, $k = 1, \ldots, \max_j l_j$. The set of all restricted maximal chains is denoted by $\mathcal{C}_m^r(L)$.

**Proposition 4.3** Suppose $l_1 \leq l_2 \cdots \leq l_n$. The number of restricted maximal chains is given by:

$$|\mathcal{C}_m^r(L)| = (n!)^{l_1} \times ((n-1)!)^{l_2-l_1} \times \cdots \times (1!)^{l_n-l_{n-1}}$$

$$= n! (n-1)!^2 \cdots 1^{l_n}.$$

If $l_1 = l_2 = \cdots = l_n =: l$ (l-choice game), then $|\mathcal{C}_m^r(L)| = (n!)^l$. 
PROOF. Let us first prove the statement for \( l \)-choice games. Any restricted maximal chain has the form \( \{(0, \ldots, 0), (1, \ldots, 1), x_n+1, \ldots, (l, \ldots, l)\} \). Between \((k, \ldots, k)\) and \((k+1, \ldots, k+1)\), for \( k = 0, \ldots, l-1 \), there are \( n! \) maximal chains since the sublattice \([[(k, \ldots, k), (k+1, \ldots, k+1)]\) is the Boolean lattice \(2^n\). Hence, \(|C^r_n(L)| = (n!)^l\).

We turn to the general case. Any restricted maximal chain has the form \( \{\bot, x_1, \ldots, (l_1, l_1), x^{l_1+1}, \ldots, (l_1, l_2, \ldots l_2), x^{l_1+l_2(n-1)+1}, \ldots, (l_1, l_2, l_3, \ldots l_3), \ldots, (l_1, \ldots, l_n)\} \). In \( L \), the sublattice \([\bot, (l_1, \ldots, l_1)]\) corresponds to a \( l_1 \)-choice game with \( n \) players, hence the number of restricted maximal chains is \((n!)^l\). The sublattice \([[(l_1, \ldots, l_1), (l_1, l_2, \ldots l_2)]\) corresponds to a \((l_2 - l_1)\)-choice game with \( n-1 \) players, hence it has \((n-1)!^{l_2-l_1}\) restricted maximal chains. Continuing the process till the sublattice \([[(l_1, \ldots, l_{n-1}, l_{n-1}), (l_1, \ldots, l_n)]\) proves the formula. \( \Box \)

The marginal worth vector \( \psi^{C^r} \) associated to a restricted chain \( C_r \) is defined as before, as well as the associated additive game \( \phi^{C^r} \). Note that \( \phi^{C^r} \equiv v \) on \( C_r \) still holds. The set of additive games \( \phi^{C^r} \) for all restricted maximal chains is denoted \( \mathcal{M}(v) \).

**Definition 4.7** The Weber set is defined as the convex hull of all additive games \( \phi^{C^r} \):

\[
\mathcal{W}(v) := \text{co}(\mathcal{M}(v)).
\]

**Theorem 4.2** If a multichoice game \( v \) is convex, then any additive game in \( \mathcal{M}(v) \) is a vertex of the core:

\[
\mathcal{M}(v) = \text{Ext}(\mathcal{W}(v)) \subseteq \text{Ext}(\mathcal{C}(v)).
\]

**Proof.** Consider a restricted maximal chain \( C_r \) and its associated additive game \( \phi^{C^r} \). We know by Th. 1.1 that it is a vertex of the precore, and since \( \phi^{C^r} \) coincide with \( v \) on \( C_r \), it has the property \( \phi^{C^r}(k \wedge l_1, \ldots, k \wedge l_n) = v(k \wedge l_1, \ldots, k \wedge l_n), k = 1, \ldots, \max_j l_j \), hence it belongs to the core and is a vertex of it. \( \Box \)

As a corollary of Th. 4.1 and 4.2 we obtain:

**Corollary 4.1** Let \( v \) be a convex multichoice game. Then

\[ (i) \quad \mathcal{PW}(v) \subseteq \mathcal{PC}^F(v) \]
\[ (ii) \quad \mathcal{W}(v) \subseteq \mathcal{C}(v) \]

Our aim is now to study the inclusion of the core in the Weber set, like in the classical case.

**Theorem 4.3** For any multichoice game \( v \), the polytope of the precore is included in the pre-Weber set, i.e., \( \mathcal{PC}^F(v) \subseteq \mathcal{PW}(v) \).

The proof relies on the following lemmas.

**Lemma 4.2** Let \( \phi, y \) be two vectors in \( \mathbb{R}^{\sum_j l_j} \), and \( \phi \) be considered as an additive game. Then their inner product writes:

\[
\phi \cdot y = \sum_{i=1}^{\sum_j l_j} \phi(x^i)(Y_{\sigma(i)} - Y_{\sigma(i+1)})
\]

for any maximal chain \( \{x^0 := \bot, x^1, x^2, \ldots, \top\} \) and corresponding mapping \( \sigma \), and \( Y_{\tilde{k}_j} := \sum_{p=k}^{l_j} y_{\tilde{p} j} \), for all \( \tilde{k}_j \in \tilde{L} \), and \( Y_{\sigma(\sum_j l_j+1)} := 0 \).
Proof. We have, letting $\sigma(i) := \widehat{k(i)}_{j(i)}$:

$$\sum_{i=1}^{l_j} \phi(x^i)(Y_{\sigma(i)} - Y_{\sigma(i+1)}) = \sum_{i=1}^{l_j} Y_{\sigma(i)}(\phi(x^i) - \phi(x^{i-1}))$$

$$= \sum_{i=1}^{l_j} \left( \sum_{p=k(i)}^{l(i)} y_{P(j)} \right) \left( \phi(\widehat{k(i)}_{j(i)}) - \phi((\widehat{k(i)}_{j(i)}) - 1) \right) \text{ by additivity of } \phi$$

$$= \sum_{i=1}^{l_j} \phi(\widehat{k(i)}_{j(i)}) \left( \sum_{p=k(i)}^{l(i)} y_{P(j)} - \sum_{p=k(i)+1}^{l(i)} y_{P(j)} \right)$$

$$= \sum_{i=1}^{l_j} \phi(\widehat{k(i)}_{j(i)}) y_{\widehat{k(i)}_{j(i)}}$$

$$= \sum_{i=1}^{l_j} \phi_{\sigma(i)} y_{\sigma(i)} = \phi \cdot y.$$

\(\square\)

Lemma 4.3 Given any \(y \in \mathbb{R}^{\sum_{j} l_j}\), there exists a maximal chain with corresponding mapping \(\sigma\) such that \(Y_{\sigma(1)} \geq \cdots \geq Y_{\sigma(\sum_{j} l_j)}\).

Proof. We partition \(\{1, \ldots, \sum_{j} l_j\}\) into \(n\) disjoint intervals \(I_i := [\sum_{j=1}^{i} l_j + 1, \sum_{j=1}^{i+1} l_j], i = 0, \ldots, n-1\), corresponding to the levels in \(L_1, \ldots, L_n\). In each interval \(I_i\), compute

$$Y_k := \sum_{p=k}^{\sum_{j=1}^{i+1} l_j} y_p, \quad k \in I_i.$$ 

Take a permutation \(\pi\) such that \(Y_{\pi(1)} \geq \cdots \geq Y_{\pi(\sum_{j} l_j)}\). Then if \(\pi(k) \in I_i\), put \(\sigma(k) := \widehat{p_k}\), where \(p\) is the first level in \(I_i\) not already chosen, and put \(Y_{\sigma(k)} := Y_{\pi(k)}\). Then by construction \(\sigma\) is compatible with a chain, and \(Y_{\sigma(i)}\) is non increasing with \(i\).

\(\square\)

Example 4.2 We illustrate Lemma 4.3. Let \(n = 2\) and \(l_1 = 2, l_2 = 3\), and consider \(y = (0.1, 0.7, 0.5, 0.3, 0.2)\). The partition of \(\{1, 2, 3, 4, 5\}\) gives \(\{\{1, 2\}, \{3, 4, 5\}\}\). Hence the computation of \(Y\) gives \(Y = (0.8, 0.7, 1, 0.5, 0.2)\), and we get:

$$\sigma(1) = 1_2, \quad \sigma(2) = 1_1, \quad \sigma(3) = 2_1, \quad \sigma(4) = 2_2, \quad \sigma(5) = 3_2,$$

which defines the maximal chain \(\{(0, 1), (1, 1), (2, 1), (2, 2), (2, 3)\}\). We have effectively \(Y_{\sigma(1)} = Y_{1_2} = 0.5 + 0.3 + 0.2 = 1, Y_{\sigma(2)} = Y_{1_1} = 0.1 + 0.7 = 0.8, \) etc.

Let us now prove Th. 4.3. Our proof uses the separation theorem for closed convex sets, similarly to the proof of Derks for proving the inclusion of the (classical) core into the Weber set [5].

Proof. Suppose it exists \(\phi\) an additive game in \(\mathcal{P}C^F(v)\), which is convex, but not belonging to the pre-Weber set. By the separation theorem, it should exists \(y \in \mathbb{R}^{\sum_{j} l_j}\) such that for any additive game \(\phi'\) in the pre-Weber set, \(\phi \cdot y < \phi' \cdot y\).

Let us consider such a \(y\) and \(\phi\). Then, for any maximal chain and corresponding mapping \(\sigma\), using Lemma 4.2 we get:

$$\phi \cdot y = \sum_{i=1}^{\sum_{j} l_j} \phi(x^i)(Y_{\sigma(i)} - Y_{\sigma(i+1)}).$$

Let us choose a maximal chain \(C\) with mapping \(\sigma\) such that \(Y_{\sigma(i)} - Y_{\sigma(i+1)} \geq 0, i = 1, \ldots, \sum_{j} l_j\). This is always possible by Lemma 4.3. Considering that \(\phi\) is in the precore, and that the additive game \(\phi^C\) in
the pre-Weber set satisfies $\phi^C = v$ on $C$, we deduce:

$$\phi \cdot y \geq \sum_{i=1}^{l_j} v(x^i)(Y_{\sigma(i)} - Y_{\sigma(i+1)})$$

$$= \sum_{i=1}^{l_j} \phi^C(x^i)(Y_{\sigma(i)} - Y_{\sigma(i+1)}),$$

a contradiction. Hence, $PC^F(v)$ is included in $PW(v)$. \hfill \triangleleft

**Theorem 4.4** For any multichoice game $v$, the core is included in the Weber set, i.e., $C(v) \subseteq W(v)$.

We need the following lemma.

**Lemma 4.4** Given any $y \in \mathbb{R}^{\sum_j l_j}$, there exists a restricted maximal chain $\{\bot, x^1, x^2, \ldots, \top\}$ with corresponding mapping $\sigma$ such that $Y_{\sigma(i)} - Y_{\sigma(i+1)} \geq 0$, for each $i \in \{1, \ldots, \sum_j l_j\}$ such that $x^i$ and $x^{i+1}$ belongs to the sublattice $[(k-1) \wedge l_1, \ldots, (k-1) \wedge l_n], (k \wedge l_1, \ldots, k \wedge l_n)$, $x^i \neq ((k-1) \wedge l_1, \ldots, (k-1) \wedge l_n)$ for some $k \in \{1, \ldots, \max_j l_j\}$.

**Proof.** Given $y$, by Lemma 4.3, there exists a maximal chain $C$ in $L$ with mapping $\sigma$ such that $Y$ is non increasing along it. It is easy to build a restricted maximal chain $C_r$ from $C$ so that the above condition is satisfied: starting from $\bot$, put $\sigma_r(1) := 1 = \sigma(1)$, then $\sigma_r(2) = 1'$ if $\sigma^{-1}(1') < \sigma^{-1}(1_k)$ for all $k \neq j, j'$, and so on till $C_r$ is defined in the first sublattice $[\bot, (1, 1, \ldots, 1)]$. Then from $(1, 1, \ldots, 1)$ proceed similarly, i.e., $\sigma_r(n+1) = 2$ if $\sigma^{-1}(2) < \sigma^{-1}(2_k)$ for all $k \neq j$, etc. \hfill \triangleleft

**Example 4.3** Let $n = 2$, $l_1 = 3$, $l_2 = 4$, and consider $y$ defined below.

<table>
<thead>
<tr>
<th>$k_i$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$l_4$</th>
<th>$l_5$</th>
<th>$l_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{ki}$</td>
<td>$0.1$</td>
<td>$0.6$</td>
<td>$0.1$</td>
<td>$0.3$</td>
<td>$0.5$</td>
<td>$0.7$</td>
</tr>
<tr>
<td>$Y_{ki}$</td>
<td>$0.8$</td>
<td>$0.7$</td>
<td>$0.1$</td>
<td>$1.7$</td>
<td>$1.4$</td>
<td>$0.9$</td>
</tr>
</tbody>
</table>

Then the corresponding maximal chain is defined by:

$$\sigma(1) = 1_2, \quad \sigma(2) = 2_2, \quad \sigma(3) = 3_2, \quad \sigma(4) = 1_2, \quad \sigma(5) = 2_2, \quad \sigma(6) = 4_2, \quad \sigma(7) = 5_2.$$

Applying Lemma 4.4, we find the following maximal restricted chain:

$$\sigma_r(1) = 1_2, \quad \sigma_r(2) = 1_2, \quad \sigma_r(3) = 2_2, \quad \sigma_r(4) = 4_2, \quad \sigma_r(5) = 5_2, \quad \sigma_r(6) = 5_2, \quad \sigma_r(7) = 4_2.$$

One can verify that

$$Y_{1_2} = 1.7 \geq Y_{1_2} = 0.8,$$

$$Y_{2_2} = 1.4 \geq Y_{2_2} = 0.7,$$

$$Y_{3_2} = 0.9 \geq Y_{3_2} = 0.1,$$

$$Y_{4_2} = 0.2,$$

as expected.

**Proof.** (of Th. 4.4) We use the same technique as for Th. 4.3. Assume $\phi \in C(v) \setminus W(v)$. Then it should exist $y \in \mathbb{R}^{\sum_j l_j}$ such that for any element $\phi'$ in the Weber set, $\phi \cdot y < \phi' \cdot y$. Considering such a $y$, we know by Lemma 4.4 that there exists a restricted maximal chain $C_r$ with corresponding mapping $\sigma_r$ such that $Y$ is non decreasing on each sublattice $[(k-1) \wedge l_1, \ldots, (k-1) \wedge l_n], (k \wedge l_1, \ldots, k \wedge l_n)$, except its bottom element $((k-1) \wedge l_1, \ldots, (k-1) \wedge l_n)$. Let us decompose $\phi \cdot y$ as follows:

$$\phi \cdot y = \sum_{i=1}^{l_j} y_i \cdot \phi_i = \sum_{k=1}^{\max_j l_j} n(k) \phi_{\sum_{j=1}^{k-1} n(j)+i} \cdot \sum_{i=1}^{\sum_{j=1}^{k-1} n(j)+i} \phi_{\sum_{j=1}^{k-1} n(j)+i}.$$
where \( n(k) = |\{ j \mid k \land l_j = k \}| \), i.e., we have decomposed the sum according to the different levels. In each term of the right hand, we apply Lemma 12. This gives:

\[
\phi \cdot y = \sum_{k=1}^{l_j} \left[ \sum_{i=1}^{n(k)-1} \phi(x^{n(k)+i})(Y_{\sigma_k(\sum_{j=1}^{k-1} n(j))} - Y_{\sigma_k(\sum_{j=1}^{n(k)-1} n(j))}) + \phi(x^{n(k)+i})(Y_{\sigma_k(\sum_{j=1}^{k-1} n(j))} - Y_{\sigma_k(\sum_{j=1}^{n(k))}}} \right].
\]

Observe that \( x^{n(k)+i} = (k \land l_1, \ldots, k \land l_n) \) by definition of \( C_r \). Since \( \phi \in C(v) \) implies that \( \phi(k \land l_1, \ldots, k \land l_n) = v(k \land l_1, \ldots, k \land l_n) \), and by Lemma 14 it follows that

\[
\phi \cdot y \geq \sum_{k=1}^{l_j} \left[ \sum_{i=1}^{n(k)-1} v(x^{n(k)+i})(Y_{\sigma_k(\sum_{j=1}^{k-1} n(j))} - Y_{\sigma_k(\sum_{j=1}^{n(k)-1} n(j))}) + v(x^{n(k)+i})(Y_{\sigma_k(\sum_{j=1}^{k-1} n(j))} - Y_{\sigma_k(\sum_{j=1}^{n(k))}}) \right]
\]

where \( \phi^{C_r} \) is the additive game corresponding to \( C_r \), for which \( \phi^{C_r} \equiv \phi \) on \( C_r \). But this contradicts the fact that \( \phi^{C_r} \) is element of the Weber set. Hence, \( \phi \in \mathcal{W}(v) \).

We are now in position to give the main result of the paper.

**Theorem 4.5** For any convex multichoice game \( v \), the following holds.

(i) \( C(v) = \mathcal{W}(v) \), or equivalently \( \text{Ext}(C(v)) = \mathcal{M}(v) \)

(ii) \( \mathcal{P}^C(v) = \mathcal{P}^W(v) \), or equivalently \( \text{Ext}(\mathcal{P}(v)) = \mathcal{P} \mathcal{M}(v) \).

**Proof.** By Th. 12 we know that for a convex game \( v \), any vertex of the Weber set is a vertex of the core. Since these are convex sets, and since the core is included in the Weber set by Th. 14, it follows that the vertices of the two sets coincide. For the precore, using Th. 13 and 15 similarly proves the result.

Fig. 1 summarizes most of the results on the core and the Weber set.

![Figure 1: Relations between the core and the Weber set: general case (left), convex case (right)](image)

**5. Comparison with previous works** Previous works on the core of multichoice games have been done mainly by Tijs et al. [14], and Faigle [7]. We summarize their approach and make a comparison with ours.

**5.1 Core of Tijs** Tijs et al. define the core of multichoice game as follows. Let \( v \) be a multichoice game on \( L \), we introduce \( M := \{ (i, k) \mid i \in N, k \in L_i \} \). Note that \( (i, k) \) corresponds to our \( k_i \). A (level)
payoff vector on $L$ is a function $\delta : M \to \mathbb{R}$, where, for all $i \in N$ and $k \in L_i \setminus \{0\}$, $\delta(i, k)$ denotes the increase in worth for player $i$ corresponding to a change of activity from level $k - 1$ to level $k$ by this player, and $\delta(i, 0) = 0$ for all $i \in N$.

To each $\delta$ we can associate bijectively an additive game $\phi_\delta$ in $\mathcal{A}(L)$ by

$$\phi_\delta(x) := \sum_{i \in N} \sum_{k=0}^{x_i} \delta(i, k), \quad \forall x \in L$$

(see Prop. 3.1 above). Hence, in the sequel we will use indifferently the payoff vector $\delta$ or its corresponding additive game $\phi_\delta$.

Let $S \subseteq N$. A payoff vector $\delta$ on $L$ is called efficient for $v$ if $\phi_\delta(\top) = v(\top)$, and it is called level increase rational for $v$ if, for all $i \in N$ and $k \in L_i \setminus \{0\}$, $\delta(i, k)$ is at least the increase in worth that player $i$ can obtain when he works alone and changes his activity from level $k - 1$ to level $k$, i.e.

$$\delta(i, k) \geq v(0_{-i}, k_i) - v(0_{-i}, (k - 1)_i).$$

**Definition 5.1** A payoff vector on $L$ is an imputation of $v$ if it is efficient and level increase rational for $v$.

We denote the set of imputations of the games $v$ by $\mathcal{I}(v)$.

**Definition 5.2** The core $\mathcal{C}_{\text{Tij}}(v)$ of the game $v$ consists of all $\delta \in \mathcal{I}(v)$ that satisfy $\phi_\delta(x) \geq v(x)$ for all $x \in L$.

Let $\phi_\delta \in \mathcal{C}_{\text{Tij}}(v)$, then it satisfies $\phi_\delta(x) \geq v(x)$, $\phi_\delta(\top) = v(\top)$. Therefore $\phi_\delta \in \mathcal{P}(v)$, which proves that $\mathcal{C}_{\text{Tij}}(v) \subseteq \mathcal{P}(v)$.

**Definition 5.3** For a multichoice game $v$, the set $\mathcal{C}_{\text{min}}(v)$ of minimal core elements is the set of least elements of the core:

$$\mathcal{C}_{\text{min}}(v) := \{ \phi_\delta \in \mathcal{C}_{\text{Tij}}(v) \mid \text{there is no } \phi_{\delta'} \in \mathcal{C}_{\text{Tij}}(v) \text{ such that } \delta' \neq \delta \text{ and } \phi_{\delta'}(x) \leq \phi_\delta(x), \forall x \in L \}. $$

Let $\mathcal{L}(v) = \{ \phi \in \mathcal{A}(L) \mid \phi(0_{-i}, k_i) - \phi(0_{-i}, (k - 1)_i) \geq v(0_{-i}, k_i) - v(0_{-i}, (k - 1)_i), k = 1, \ldots, l_i \}$. We have clearly: $\mathcal{P}(v) \cap \mathcal{L}(v) = \mathcal{C}_{\text{Tij}}(v)$.

In summary, we have the following inclusions.

**Proposition 5.1** For any multichoice game $v$ on $L$, the following holds.

$$\mathcal{C}(v) \cap \mathcal{L}(v) \subseteq \mathcal{C}_{\text{Tij}}(v) = \mathcal{P}(v) \cap \mathcal{L}(v) \subseteq \mathcal{P}(v).$$

Tij et al. define the Weber set as follows. Define $M^+ := M \setminus \{(i, 0), i \in N\}$, which coincides with our $\bar{L}$. An admissible ordering (for $v$) is a bijection $\pi : M^+ \to \{1, \ldots, \sum_{i \in N} l_i \}$ satisfying

$$\pi(i, k) < \pi(i, k + 1), \quad i \in N, \quad k = 1, \ldots, l_i - 1.$$ 

This amounts to define a maximal chain $\{1, x^{+1}, \ldots, x^{\sum_{i \in N} l_i}\} \subseteq L$, where $x^{k}_\pi$ is the element of $L$ obtained after $k$ steps according to $\pi$. The marginal vector $w^{\pi}$ associated to $\pi$ is defined as

$$w^{\pi}_{i, k} = v(x^{\pi(i, k)}_\pi) - v(x^{\pi(i, k) - 1}_\pi), \quad (i, k) \in M^+.$$ 

Considering $w^{\pi}$ as a payoff vector, we can associate the additive game $\phi_{w^{\pi}}$. The payoff vector is an imputation if the game is monotonic after zero-normalization (i.e., such that $v(0_{-i}, k_i) = 0$, for all $i, k$). Then the Weber set in the sense of Tij is defined as the convex hull of all marginal vectors.

In fact, $\phi_{w^{\pi}}$ is exactly our additive game $\phi^C$, where $C$ is the maximal chain induced by $\pi$. Hence, $\mathcal{W}_{\text{Tij}}(v) = \mathcal{P}(v)$.

Tij et al. showed that if $v$ is convex, then $\mathcal{W}_{\text{Tij}}(v)$ is the convex hull of $\mathcal{C}_{\text{min}}$. Comparing to our results, we have proven that $\mathcal{P}(v)$ is the convex hull of the extreme points of $\mathcal{P}(v)$. This shows that the extreme points of $\mathcal{P}(v)$ are all minimal core elements. However, not all minimal core elements are extreme points as the following example shows.
Example 5.1 (borrowed from [14, Ex. 3]) Let us consider a game with 2 players, \( l_1 = 2, l_2 = 1 \), and a convex game \( v \) defined as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( v(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>0</td>
</tr>
<tr>
<td>(1,0)</td>
<td>0</td>
</tr>
<tr>
<td>(2,0)</td>
<td>0</td>
</tr>
<tr>
<td>(0,1)</td>
<td>2</td>
</tr>
<tr>
<td>(1,1)</td>
<td>2</td>
</tr>
<tr>
<td>(2,1)</td>
<td>3</td>
</tr>
</tbody>
</table>

There are 3 maximal chains: \( C_1 := \{(0,0), (1,0), (2,0), (2,1)\} \), \( C_2 := \{(0,0), (1,0), (1,1), (2,1)\} \) and \( C_3 := \{(0,0), (0,1), (1,1), (2,1)\} \), leading to the three following additive games \( \phi_1, \phi_2, \phi_3 \):

\[
\begin{array}{ccc}
\phi_1(1,0) = 0 & \phi_2(1,0) = 0 & \phi_3(1,0) = 2 \\
\phi_1(2,0) = 0 & \phi_2(2,0) = 2 & \phi_3(2,0) = 3 \\
\phi_1(0,1) = 3 & \phi_2(0,1) = 2 & \phi_3(0,1) = 0 \\
\end{array}
\]

Clearly, these are all minimal core elements as expected. But the following additive game is a minimal core element:

\( \phi^4(1,0) = 1, \ \phi^4(2,0) = 2, \ \phi^4(0,1) = 1 \)

and is not among the three additive games of \( PM(v) \).

This shows that \( C_{\text{min}} \) is unnecessarily large, since it contains elements which are not extreme points. Finally, we prove the following.

Proposition 5.2 For any convex game \( v \), \( PC^F(v) \subseteq \mathcal{C}_{\text{Fij}}(v) \). In other words, any convex combination of chain vectors is level increase rational for a convex game \( v \).

Proof. Consider \( \phi^C \) in \( PM(v) \), for some maximal chain \( C = \{1, x^1, \ldots, x^{n-1}, l_j = \top \} \). Using previous notations, the associated chain vector is \( \psi^C \), with

\[
\psi^C_i = v(x^i) - v(x^{i-1}), \quad i = 1, \ldots, \sum_{j=1}^n l_j.
\]

Consider \( i_j \) in \( 1, \ldots, \sum_{j=1}^n l_j \) such that \( x_i^j = k \) and \( x_i^{j-1} = k - 1 \) for some level \( k \). By convexity of \( v \), we have:

\[
\psi^C_{i_j} = v(x^{i_j}) - v(x^{i_j-1}) \geq v(0_{-i_1,k_i}) - v(0_{-i_1}, (k-1)i_1).
\]

This being true for each \( i_j \), we have proven that the payoff vector \( \Psi^C \) is level increase rational. Since convex combinations preserves the property of level increase rationality, and since \( PC^F(v) \) is the convex hull of \( PM(v) \) by Th. 4.30 the proof is complete.

5.2 Core of Faigle U. Faigle introduced the core of games with restricted cooperation in [11]. He proposes a general model for a cooperative game on a finite set \( N \) of players, without assuming that every coalition \( S \subseteq N \) of players is feasible, and thus takes into account the situation, for instance, where some players may join a coalition only if some other players have already joined the coalition.

Definition 5.4 [11] A (finite) game with restricted cooperation is a quadruple \( \gamma = (N,F,v,v_0) \) where \( N \) is the finite set of players, \( F \) a nonempty collection of subsets of \( N \) called feasible coalitions, \( v : F \rightarrow \mathbb{R} \) the value function, with \( v(\emptyset) = 0 \), and \( v_0 \in \mathbb{R} \) the total value of the game \( \gamma \). If \( v \) and \( v_0 \) are non-negative, \( \gamma \) is a positive game.

A solution of the game \( \gamma \) is a fair distribution of its value \( v_0 \) among the players. As usual, we therefore define the core \( C_{\text{Faigle}}(v) \) of the game \( \gamma \) to consist of all undominated imputations, i.e. vectors \( \phi \in \mathbb{R}^N \) such that:

(i) \( \sum_{x \in A} \phi_x \geq v(A) \) for all \( A \in F \),

(ii) \( \sum_{x \in N} \phi_x = v_0 \).
The usual way to define the restricted set of coalitions $F$ is to introduce a partial order $\leq$ on $N$. Let $P = (N, \leq)$ denote this partially ordered set of players. The relation $i \leq j$ indicates that the presence of $j$ enforces the presence of $i$ in any coalition $S \subseteq N$. Hence, a (feasible) coalition of $P$ is a subset $S \subseteq N$ such that $s \in S$ and $t \subseteq s$ yield $t \in S$ for all $s, t \in N$. Thus unions and intersections of coalitions are again coalitions, while complements of coalitions may fail to have this property. By $\Gamma(P)$ we denote the vector space of all cooperative games on $P$.

Let $v$ be a multichoice game on $L_i$, keeping our previous notations, and consider as a set of virtual players the disjoint union $N' = \cup_{i \in N}(L_i \setminus \{0_i\})$, with the ordering $\leq$ defined by $n_1 \leq n_2$ if $n_1, n_2 \in L_i$ for some $i$ and $n_1 \leq n_2$ in $L_i$. There is a one-to-one correspondence between the participation profiles $x \in L = \prod_{i=1}^{n} L_i$ and the feasible coalitions on $N'$ in the sense of Faigle: the non-zero components of such an $x$ may be interpreted as the maximal elements of a feasible coalition of $N'$ and conversely, while the zero-vector $0 \in L$ corresponds to the empty feasible coalition of $N'$. Hence any multichoice game can be considered as a particular game with precedence constraints, on a virtual set of players $N'$.

Let $x = (x_1, \ldots, x_n) \in L \setminus \{\bot\}$, and consider $S := \{s \in N' \mid \exists i \in N, \text{ s.t. } s \leq x_i\}$. Then $S$ is a feasible coalition of $N'$. Taking any $\phi \in \mathbb{R}^{N'}$ belonging to $\mathcal{C}_F$, we have $\sum_{s \in S} \phi_s \leq v(x)$ and $\sum_{s \in N'} \phi_s = v(\top)$, which entails that the corresponding additive game $\Phi$ belongs to $\mathcal{PC}(v)$, and reciprocally. This proves $\mathcal{PC}(v) = \mathcal{C}_F(v)$.

References