1 Introduction

The concept of bi-capacity has recently been proposed by Grabisch and Labreuche [7, 5] as a generalization of capacities (or fuzzy measures) in the context of decision making. Specifically, let us consider a set $X$ of alternatives in a multicriteria decision making problem, where each alternative is described by a set of $n$ real valued scores $(a_1, \ldots, a_n)$. Suppose one wants to compute a global score of this alternative by the Choquet integral w.r.t. a capacity $\mu$, namely $C_{\mu}(a_1, \ldots, a_n)$. Then it is well known that the correspondence between the capacity and the Choquet integral is $\mu(A) = C_{\mu}(1_A, 0_A^c)$, $\forall A \subseteq N$, where $(1_A, 0_A^c)$ is an alternative having 1 as score on all criteria in $A$, and 0 otherwise. Such an alternative is called a binary alternative, and the above result says that the capacity represents the overall score of all binary alternatives.

However, in many practical situations, it is suitable to score alternatives on a bipolar scale, i.e. with a central value 0 having the meaning of a borderline between positive scores, considered as good, and negative scores, considered as bad. It has been observed that most often human decision makers have a different behaviour when faced with alternatives having positive and negative scores, which means that a decision model based solely on the classical Choquet integral, hence on binary alternatives, is no more sufficient. One should, in the general case, consider all ternary alternatives, i.e. alternatives of the form $(1_A, -1_B, 0_{(A \cup B)^c})$. Clearly we need two arguments to denote the overall score of ternary alternatives, namely $v(A, B)$, with $A, B \subseteq N$ being disjoint. This defines bi-capacities, by analogy with capacities.

An interesting question is to define for bi-capacities the concept of Möbius transform and interaction, since they are very useful in applications. This was done in [5], leading to a Möbius transform and an interaction index with two disjoint sets as arguments. Due to the complexity of bi-capacities (they require $3^n$ values to be defined), it is important to derive computations between the three possible representations of a bi-capacity, namely $v$ itself, its Möbius transform and its interaction index, as it was done for capacities [3]. The aim of this paper is precisely to provide a general framework for these computations, in the spirit of [3].

Throughout the paper, the cardinal of a set is denoted by the corresponding small letter, e.g. $|N| = n$.

2 Background

We introduce necessary concepts for the sequel. We consider a finite set $N := \{1, \ldots, n\}$ which can be thought as the set of criteria, states of nature, voters, etc. We denote $\mathcal{P} := \mathcal{P}(N)$.

A capacity $v : \mathcal{P} \to [0, 1]$ is a set function satisfying $v(\emptyset) = 0$ and $A \subseteq B$ implies $v(A) \leq v(B)$. If in addition $v(N) = 1$, $v$ is said normalized. Unanimity games $u_C$, $C \subseteq N$, are particular capacities — except for $C = \emptyset$ — defined by:

$$u_C(A) := \left\{ \begin{array}{ll} 1 & \text{if } A \supseteq C \\ 0 & \text{otherwise} \end{array} \right., A \subseteq N.$$  

The Möbius transform of a capacity is
defined by
\[ m^v(A) := \sum_{C \subseteq A} (-1)^{a-c} v(C), \]
for all \( A \subseteq N \). Any set function \( v \) can be represented in the basis of unanimity games:
\[ v(A) = \sum_{C \in P} m^v(C) u_C(A), \quad A \subseteq N. \]
The Möbius transform represents the coordinates of \( v \) in the basis of unanimity games.

To introduce the interaction index, we need to define the notion of derivative of a capacity: for any \( S \) belonging to \( P \setminus \{ \emptyset \} \), the derivative with respect to \( S \) of \( v \) at point \( K \in P(N \setminus S) \) is given by [5]:
\[ \Delta_S v(K) := \sum_{S' \subseteq S} (-1)^{s-s'} v(K \cup S'). \]
The interaction index has been proposed by Grabisch [4] and expresses the interaction among a coalition (group) \( C \subseteq N \) of elements:
\[ I^v(S) := \sum_{K \subseteq N \setminus S} \frac{(n-k-s)!}{(n-s+1)!} \Delta_S v(K). \]
This definition extends in fact the Shapley value [10] \( \phi^v \) and the interaction index \( I_{ij} \) for a pair of elements \( i, j \) in \( N \), introduced by Murofushi and Soneda [9]. In particular, the Shapley value is defined by
\[ \phi^v_i := \sum_{K \subseteq N \setminus i} \frac{(n-k-1)!}{n!} \Delta_i v(K), \quad i \in N. \]
We have \( I^v(\{i\}) = \phi^v_i \).

As said in the introduction, a natural generalisation is to enrich this model by fixing the overall score of all ternary alternatives \((1_A, -1_B, 0_{A \cup B})^c\), for all disjoint pairs \((A, B)\) in \( P^2 \). Let us denote \( v(A, B) \) the overall score of \((1_A, -1_B, 0_{A \cup B})^c\). We denote
\[ Q = Q(N) := \{ (A, B) \in P \times P \mid A \cap B = \emptyset \}. \]

**Definition 1** A function \( v : Q \to \mathbb{R} \) is a bi-capacity if it satisfies:

(i) \( v(\emptyset, \emptyset) = 0 \),

(ii) \( A \subseteq B \) implies \( v(A, \cdot) \leq v(B, \cdot) \) and \( v(\cdot, A) \geq v(\cdot, B) \).

In addition, \( v \) is normalized if \( v(N, \emptyset) = 1 = -v(\emptyset, N) \).

We give some insight into the structure of \( Q \), which is of primary importance. It is easy to see that \((Q, \sqsubseteq)\) is the lattice \( 3^N \), with \((A, B) \sqsubseteq (C, D)\) if \( A \subseteq C \) and \( B \supseteq D \). Supremum and infimum are respectively
\[ (A, B) \sqcup (C, D) = (A \cup C, B \cap D), \]
\[ (A, B) \sqcap (C, D) = (A \cap C, B \cup D), \]
\[ (A, B), (C, D) \in Q. \]

Top and bottom of \( Q \) are respectively denoted by \( \top \) and \( \bot \). We give as an illustration \((Q, \sqsubseteq)\) for \( n = 3 \) in Fig. 1. An important fact is that every element of \( Q \) can be obtained as a supremum over a subset of
\[ \{ (\emptyset, i^c), (i, i^c), \forall i \in N \}, \]
the set of join-irreducible elements (which are represented by black circles, in Fig. 1). Specifically, for any \((A, B) \in Q\), we have [1]:
\[ (A, B) = \bigcup_{i \in A} (i, i^c) \sqcup \bigcup_{j \in N \setminus B} (\emptyset, j^c). \]

This permits to define layers in \( Q \) as follows: for \( k \in N_0 := \{0, \ldots, n\} \), layer \( k \) contains all elements \((A, B)\) such that \(|B| = n - k\). So we denote by \( v \cdot v \) the function which links every element of \( Q \) the layer to which it belongs.

The Möbius transform of \( v \) is expressed by
\[ m^v(A, B) := \sum_{D \cap A = \emptyset} (-1)^{a-c+d-b} v(C, D), \]
\[ (A, B) \in Q. \]

The inverse equation is
\[ v(A, B) = \sum_{D \subseteq (C, D) \subseteq (A, B)} m^v(C, D), \]
\[ (A, B) \in Q. \]

We extend the notion of derivative of a set function to bi-capacities (in fact to any function on \( Q \)). As bi-capacities are defined on \( Q \), so should be the variables used in derivation. For any \( i \in N \), the derivatives with respect to any join-irreducible elements \((i, i^c)\) and \((\emptyset, i^c)\) of \( v \) at point \((K, L)\) are given by [8]:
\[ \forall (K, L) \in Q(N \setminus i) \]
\[ \Delta_{(i, i^c)} v(K, L) := v(K \cup i, L) - v(K, L), \]
\[ \Delta_{(\emptyset, i^c)} v(K, L) := v(K, L \setminus i) - v(K, L). \]
The monotonicity of $v$ entails that these derivatives are non negative. Higher order derivatives can be defined recursively by:

$$
\Delta_{(S,T)}v(K,L) := \Delta_{(i,j)}(\Delta_{(S\setminus T,U)}v(K,L))
= \Delta_{(0,0)}(\Delta_{(S,T,U)}v(K,L)),
(S,T) \in \Omega \setminus \{(0,0)\}.
$$

Since bi-capacities are defined on $\Omega$, the interaction index should be characterized for all bi-coalitions in $\Omega$. The most natural definition seems to use the derivative, as for the classical case. We propose the following

**Definition 2** Let $(S,T) \in \Omega$. The interaction index of $v$ w.r.t. $(S,T)$ is defined by:

$$
I^v(S,T) := \sum_{K \subseteq T} \frac{(t-k)!k!}{(t+1)!} \Delta_{(S,T)}v(K, (K \cup S)^c).
$$

Equation (4)

3 Operators on $\Omega \times \Omega$

In [3], the authors introduced some operators defined on the power set $\mathcal{P}$, which enable the writing of the formulae seen in the last section under a simplified algebraic form. This provided a convenient framework for expressing various transformations between capacities, their Möbius transform and interaction index. In the first place, we recall main definitions. Considering real functions on $\mathcal{P}$ in one (capacities) and two (operators) variables, a multiplication $\ast$ is introduced between operators, and between a capacity and an operator, as follows: let $v$ be a capacity and $\Phi, \Psi$ some operators; for $A_1, A_2$ belonging to $\mathcal{P}$, we have:

$$(\Phi \ast \Psi)(A_1, A_2) := \sum_{C \in \mathcal{P}} \Phi(A_1, C) \Psi(C, A_2),$$

$$(\Phi \ast v)(A_1) := \sum_{C \in \mathcal{P}} \Phi(A_1, C) v(C),$$

$$(v \ast \Psi)(A_2) := \sum_{C \in \mathcal{P}} v(C) \Phi(C, A_2).$$

Let us consider a subset $\mathcal{G}_\mathcal{P}$ of these operators\(^1\), defined by the operators $\Phi$ which have the property

$$
\Phi(A_1, A_2) = \begin{cases} 
1 & \text{if } A_1 = A_2 \\
0 & \text{if } A_1 \not\subseteq A_2,
\end{cases}
$$

for any $A_1, A_2 \in \mathcal{P}$. In the set $\mathcal{G}_\mathcal{P}$, we found the operators Zeta and Möbius for capacities (cf. [6]).

We will proceed in the same way for bi-capacities, the basis working set being $\Omega$. We consider real functions on $\Omega$ in one (bi-capacities) and two (operators) variables: we introduce a multiplication $\ast$ between operators, and between a bi-capacity and an operator. Let $v$ be a bi-capacity and $\Phi, \Psi$ some operators; for $(A_1, B_1), (A_2, B_2)$ belonging to $\Omega$, we define:

$$(\Phi \ast \Psi)((A_1, B_1), (A_2, B_2)) := \sum_{(C,D) \in \Omega} \Phi((A_1, B_1), (C, D)) \Psi((C, D), (A_2, B_2)),$$

$$(\Phi \ast v)((A_1, B_1)) := \sum_{(C,D) \in \Omega} \Phi((A_1, B_1), (C, D)) v(C, D),$$

$$(v \ast \Psi)((A_2, B_2)) := \sum_{(C,D) \in \Omega} v(C, D) \Phi((C, D), (A_2, B_2)).$$

\(^1\)The sets and functions denoted with the suffix $\mathcal{P}$ are sets and functions defined on $\mathcal{P}$ referring to [3].
Endowed with $\ast$, the set of these operators contains the neutral element $\Delta$ defined by
\[
\Delta((A_1, B_1), (A_2, B_2)) := \\
\begin{cases} 
1 & \text{if } (A_1, B_1) = (A_2, B_2) \\
0 & \text{else},
\end{cases}
\]
for $(A_1, B_1), (A_2, B_2)$ belonging to $\mathcal{Q}$ and verifies associativity. When it exists, we will denote $\Phi^{-1}$ the inverse of an operator $\Phi$, that is to say the operator verifying $\Phi \ast \Phi^{-1} = \Phi^{-1} \ast \Phi = \Delta$.

The following proposition deals with a subset of the set of operators which is important for our study.

**Proposition 1** The family $\mathcal{G}$ of operators defined by:
\[
\Phi \in \mathcal{G} \iff \forall (A_1, B_1), (A_2, B_2) \in \mathcal{Q} : \\
\Phi((A_1, B_1), (A_2, B_2)) = \\
\begin{cases} 
1 & \text{if } (A_1, B_1) = (A_2, B_2) \\
0 & \text{if } (A_1, B_1) \not\subseteq (A_2, B_2)
\end{cases}
\]
provided with the operation $\ast$ is a group. The inverse $\Phi^{-1}$ of $\Phi$ in $\mathcal{G}$ computes recursively through
\[
\Phi^{-1}((A_1, B_1), (A_1, B_1)) = 1, \\
\Phi^{-1}((A_1, B_1), (A_2, B_2)) = - \sum_{(C, D) \in [(A_1, B_1), (A_2, B_2) \in \mathcal{Q}]} \Phi^{-1}((A_1, B_1), (C, D)) \Phi((C, D), (A_2, B_2)).
\]
We can find among the set $\mathcal{G}$ the operator $Z$ (Zeta) through which we write the relation (3):
\[
v = m^v \ast Z
\]
where, for every $(A_1, B_1), (A_2, B_2) \in \mathcal{Q} :
\[Z((A_1, B_1), (A_2, B_2)) := \\
\begin{cases} 
1 & \text{if } (A_1, B_1) \subseteq (A_2, B_2) \\
0 & \text{otherwise}.
\end{cases}
\]
Similarly, the Möbius operator (inverse of $Z$) verifies
\[
m^v = v \ast Z^{-1} \quad \text{(formula (2))}
\]
where Proposition 1 gives
\[
Z^{-1}((A_1, B_1), (A_2, B_2)) = \\
\begin{cases} 
(-1)^{a_2 - a_1 + b_1 - b_2} & \text{if } (A_1, B_1) \subseteq (A_2, B_2) \\
0 & \text{else},
\end{cases}
\]
$(A_1, B_1), (A_2, B_2) \in \mathcal{Q}$.

In [3], the interaction index of a capacity was expressed through a $\mathcal{G}_p$ operator, which facilitated the inversion of $I$. We shall undertake to do the same thing for bi-capacities. From formula (4) and thanks to an expression of the derivatives based on Möbius transform [5], we have for every $(S, T) \in \mathcal{Q}$:
\[
I^v(S, T) = \sum_{(S', T') \in [(S, T), (S \cup T, \emptyset)]} \frac{1}{t - t' + 1} m^v(S', T').
\]
As a result, if we set down:
\[
\Gamma((A_1, B_1), (A_2, B_2)) := \\
\begin{cases} 
\frac{1}{b_1 - b_2 + 1} & \text{if } (A_1, B_1) \subseteq (A_2, B_2) \\
0 & \text{otherwise},
\end{cases}
\]
we can write from (5) the relation:
\[
I^v = \Gamma \ast m^v. \quad (7)
\]
Let us notice that $\Gamma$ is an operator of $\mathcal{G}$. We call it the inverse Bernoulli operator. This name will be justified in Section 6.

$\Gamma$ has a similar expression to that of $\Gamma_p$,
\[
\Gamma_p(A_1, A_2) := \\
\begin{cases} 
\frac{1}{a_2 - a_1} & \text{if } A_1 \subseteq A_2 \\
0 & \text{otherwise},
\end{cases}
\]
$((A_1, A_2) \in \mathcal{P}^2)$, with however a rather unexpected inequality $(A_2, B_2) \subseteq (A_1 \cup B_1, \emptyset)$ which will complicate the continuation of the work. Nevertheless, at this point, we can set the following fundamental result — already known in the case of capacities:

**Theorem 2** For any bi-capacity $v$, the triangular diagram where appear the functions $v, m^v, I^v$ and the operators of transition $Z, \Gamma$ is commutative.

### 4 Level operators

Our aim being now the inversion of $\Gamma$, a few results about lattice theory need to be brought in. First, the double-inequality in (6) suggests us to introduce a new binary relation on $\mathcal{Q}$.
One can give a graphic interpretation of the \( Q \) in order and the \( \leq \) Denition 3 morphic to \( N \):

\[
\begin{align*}
\Gamma \times (\star \cdot z^{-1}) & \quad \Gamma \times (z^{-1} \cdot \star) \quad \Gamma \times \star \quad I^{v} \\
\star \cdot z & \quad (z^{-1} \cdot \star) \quad z & \quad \Gamma \cdot z
\end{align*}
\]

Figure 2: Three ways of representing bi-capacities

Thus, we define the new relation \( \preceq \) on \( Q \) by:

\[
(A_1, B_1) \preceq (A_2, B_2) \quad \text{if and only if} \quad \left\{ \begin{array}{l}
(A_1, B_1) \subseteq (A_2, B_2) \\
A_2 \subseteq A_1 \cup B_1 \\
A_1 \subseteq A_2 \subseteq A_1 \cup B_1 \\
B_1 \supseteq B_2.
\end{array} \right.
\]

Next, we will denote

\[
Q_{(A,B)} := \{(C, D) \in Q \mid (C, D) \preceq (A, B)\}.
\]

As \( \preceq \) is an ordered relation included in \( \subseteq \), we have the following proposition:

**Proposition 3** For any \((A, B)\) of \( Q \), the ordered subset \((Q_{(A,B)}, \preceq)\) of \((Q, \subseteq)\) is a Boolean lattice isomorphic to \((P(B^c), \subseteq)\) by the mapping:

\[
q_{(A,B)} : Q_{(A,B)} \rightarrow P(B^c)
\]

\[
(C, D) \mapsto D^c.
\]

(8)

In particular, \((Q \uparrow, \preceq)\) is a Boolean lattice isomorphic to \((P, \subseteq)\).

Endowed with this new \( \preceq \) ordered relation, we can define the following operation in \( Q \):

**Definition 3** The strict difference operation in \( Q \) is given for every \(( (A_1, B_1), (A_2, B_2) ) \in Q \times Q \) such that \((A_1, B_1) \preceq (A_2, B_2)\) by:

\[
(A_2, B_2) \setminus (A_1, B_1) := (A_2 \setminus A_1, (B_1 \setminus B_2)^c).
\]

One can give a graphic interpretation of the \( \preceq \) order and the \( \setminus \) operation: we call vertices of \( Q \) any element \((A, B)\) such that \( A \cup B = N \), since they coincide with the vertices of \([-1, 1]^n\). In the same way, we define the vertices of any sub-lattice of \( Q \). So, for any \((A, B) \in Q\), \( Q_{(A,B)} \) is the set of the vertices of the sub-lattice \([L, (A, B)]\). Moreover, two bi-coalitions \((C_1, D_1), (C_2, D_2)\) of \( Q \) are said complementary w.r.t. an element \((A, B)\) of \( Q \) if:

\[
(A, B) \setminus (C_1, D_1) = (C_2, D_2),
\]

which is equivalent to \((A, B) \setminus (C_2, D_2) = (C_1, D_1)\).

In particular, the couples of bi-coalitions which are complementary w.r.t. \( \top \) are the couples \(\{(A, A^c), (A^c, A)\}\), for every \(A \subseteq N\). As a consequence, for \((A, B) \in Q\), complementarity w.r.t. an element of \( Q_{(A,B)} \) entails the same property that complementarity w.r.t. an element of \( P \): if \((C, D)\) belongs to \( Q_{(A,B)}\), \((C, D)\) and its complement w.r.t. \((A, B)\) are opposite vertices in the sub-lattice \( Q_{(A,B)} \).

Like the strict difference in \( P \) (cf. [3]), the strict difference operation in \( Q \) will allow us to transform some \( \mathcal{G} \) operators into operators of a single variable. In addition, the \( \star \) operation will be transformed into a convolution operation.

Once more, let us return to the case of the capacities and give results for particular operators of \( g_P \). A level operator \( \Phi \) for capacities is a \( P \times P \) function defined by

\[
\Phi(A_1, A_2) = \begin{cases} 
\Phi(\emptyset, A_2 \setminus A_1) & \text{if } A_1 \subseteq A_2 \\
0 & \text{otherwise}
\end{cases}
\]

and their set is denoted by \( g_P \). If we set down

\[
g_P := \{ \varphi : P \rightarrow \mathbb{R} \mid \varphi(\emptyset) = 1 \},
\]

we show that \( g_P \) is isomorphic to \( g_P \) by associating with any level operator \( \Phi \) the function \( \varphi^\Phi \) of \( g_P \) defined by \( \varphi^\Phi := \Phi(\emptyset, \cdot) \). Indeed, it is easy to see that \( \varphi^\Phi \) determines \( \Phi \) uniquely: let \( \varphi \) be in \( g_P \); if we define

\[
\Phi_{\varphi}(A_1, A_2) := \begin{cases} 
\varphi(A_2 \setminus A_1) & \text{for } A_1 \subseteq A_2, \\
0 & \text{else},
\end{cases}
\]

we have \( \Phi_{\varphi} = \Phi \) iff \( \varphi = \varphi^\Phi \). And if we define the operation \( \star \) between two elements \( \varphi, \psi \) of \( g_P \) by

\[
\varphi \star \psi(A) := \Phi_{\varphi} \star \Phi_{\psi}(\emptyset, A), \quad A \in P,
\]
the isomorphism is given. \( \varphi \ast \psi \) is the convolution of \( \varphi, \psi \in g \mathcal{P} \),

\[
\varphi \ast \psi(A) = \sum_{C \subseteq A} \varphi(C) \psi(A \setminus C), \quad A \in \mathcal{P}.
\]

Lastly, we write the inverse Bernoulli function for capacities \( \gamma_{\mathcal{P}} := \varphi_{\mathcal{P}} \):

\[
\gamma_{\mathcal{P}}(A) = \frac{1}{a + 1}, \quad A \in \mathcal{P}.
\]

Now, let us write corresponding results for \( \Omega \times \mathcal{Q} \) operators:

**Definition 4** A level operator \( \Phi \) is an operator in \( \mathcal{G} \) verifying for any \((A_1, B_1), (A_2, B_2)\) belonging to \( \Omega \):

\[
\Phi((A_1, B_1), (A_2, B_2)) = \left\{ \begin{array}{ll}
\Phi(\perp, (A_2, B_2) \downarrow (A_1, B_1)) & \text{if } (A_1, B_1) \subseteq (A_2, B_2), \\
0 & \text{otherwise}.
\end{array} \right.
\]

We will denote \( \mathcal{G}' \) the set of level operators.

We can notice that \( \Gamma \) is a level operator, contrary to \( Z \) and \( Z^{-1} \), even if in the case of capacities, we can find in the \( \mathcal{G}'_{\mathcal{P}} \) set the \( \Gamma_{\mathcal{P}} \) operator but also the Zeta and Möbius \( \mathcal{P} \times \mathcal{P} \) operators.

To any operator \( \Phi \), we associate a function

\[
\varphi_{\Phi}(A, B) := \Phi(\perp, (A, B)), \quad (A, B) \in \Omega.
\]

If \( \Phi \) is a level operator, then \( \varphi \) determines \( \Phi_{\varphi} := \Phi \) uniquely:

\[
\Phi_{\varphi}((A_1, B_1), (A_2, B_2)) = \left\{ \begin{array}{ll}
\varphi((A_2, B_2) \downarrow (A_1, B_1)) & \text{for } (A_1, B_1) \subseteq (A_2, B_2), \\
0 & \text{otherwise},
\end{array} \right.
\]

and one can show that \( \Phi^{-1} \) is a level operator, too. Hence, since

\[
g := \{ \varphi : \Omega \to \mathbb{R} \mid \varphi(\perp) = 1 \}
\]

is a group where we define the operation \( \ast \) by

\[
\varphi \ast \psi(A, B) := \Phi_{\varphi} \ast \Phi_{\psi}(\perp, (A, B)), \quad (A, B) \in \Omega,
\]

we show that \( \mathcal{G}' \) is isomorphic to \( g \) thanks to the mapping \( \Phi \mapsto \varphi_{\Phi} \). Moreover, \( \varphi \ast \psi \) is the convolution of \( \varphi, \psi \in g \):

\[
\varphi \ast \psi(A, B) = \sum_{(C, D) \subseteq (A, B)} \varphi(C, D) \psi((A, B) \setminus (C, D)).
\]

The convolution being a commutative operation, \( g \) is an Abelian group. The neutral element \( \Delta \) of \( \mathcal{G} \) becomes the neutral element \( \delta \) of \( g \).

\[
\delta(A, B) := \varphi_{\Delta}(A, B) = \left\{ \begin{array}{ll}
1 & \text{if } (A, B) = \perp, \\
0 & \text{otherwise}.
\end{array} \right.
\]

The inverse of \( \varphi \in g \) will be denoted \( \varphi^{-1} \) as it is common for convolutions.

As a consequence, we can express the inverse Bernoulli function \( \gamma : = \Phi_{\Gamma} \). Thanks to what have been seen before, we can directly write

\[
\gamma(A, B) = \frac{1}{n - b + 1}, \quad (A, B) \in \Omega. \quad (9)
\]

\( \gamma^{-1} \) will be the Bernoulli function (for bi-capacities).

## 5 Cardinality operators

In [3], the authors introduced the notion of cardinality function and expressed in this way the inverse Bernoulli function for capacities \( \gamma_{\mathcal{P}} \) (\( \gamma \) in [3]) under the simplified cardinal form \( f_{\gamma_{\mathcal{P}}}(a) = \frac{1}{a + 1} \).

More generally, a cardinality function on \( \mathcal{P} \) is a real function that only depends on the cardinality of the variable, and is equal to 1 in \( \emptyset \). The set of such functions will be denoted here by \( c_{\mathcal{P}} \). When \( \varphi \) is a cardinality function of \( c_{\mathcal{P}} \), it is associated with its cardinal representation \( f_{\varphi} \) in the set

\[
\tau := \{ f : N_0 \to \mathbb{R} \mid f(0) = 1 \}
\]

in a bijective way; for any \( A \in \mathcal{P} \)

\[
f_{\varphi}(|A|) = \varphi(A).
\]

Conversely, for \( f \in \tau \), we note

\[
\varphi_{f_{\mathcal{P}}}(A) = f(a), \quad A \in \mathcal{P}.
\]

Once more, \( \tau \) is an Abelian group with \( f_{\delta} := 1_{\{0\}} \) as neutral element, and where the convolution operation is, for any \( f, g \in \tau \), \( A \in \mathcal{P} \)

\[
f \ast g(|A|) := \varphi_{f_{\mathcal{P}}} \ast \varphi_{g_{\mathcal{P}}}(A),
\]

where \( \ast \) is the convolution operator.
that is to say, for any \( m \in N_0 \):

\[
f \ast g(m) = \sum_{k=0}^{m} \binom{m}{k} f(k) g(m - k).
\]

We will denote \( f^{*-1} \) the inverse of an element \( f \) of \( \tau \).

In a similar way, our last stage will be to extend this notion: a real function of \( Q \) will be called a cardinality function if it only depends on the layer of the variable and is equal to 1 for \( \perp \). We denote \( c \) the set of these functions and we will name again cardinal representation of a function \( \varphi \) of \( c \), the function \( f_\varphi \) of \( \tau \) verifying

\[
f_\varphi((A,B))) = \varphi(A,B), \quad (A,B) \in Q.
\]

Furthermore, we call cardinality operator of \( Q \times Q \) (resp. \( P \times P \)) any level operator \( \Phi \) of which the associated function \( \varphi_\Phi \) of \( g \) (resp. \( g_P \)) belongs to \( c \) (resp. \( c_P \)). Their set will be denoted by \( G'' \) (resp. \( G''_P \)). As a consequence, through transitivity, \( c_P \) and \( c \) are isomorphic, as well as \( G''_P \) and \( G'' \). Last but not least, \( (c,*) \) and \( (c_P,*) \) are subgroups of \( (g,*) \) and \( (g_P,*) \).

Therefore, by (9) the inverse Bernoulli function for bi-capacities has cardinal representation \( f_\gamma(m) = \frac{1}{m+1}, \ m \in N_0 \). In fact, it appears that \( f_\gamma = f_{\gamma_P} \). In addition, the mapping \( \varphi \mapsto f_\varphi \) is bijective; it is even an isomorphism for the groups \( c \) and \( \tau \) endowed with the operations \( \ast \). This link with the previous result is fundamental in our work.

As a conclusion to these three sections, we can give the following recapitulative result, illustrated by Figure 3:

**Proposition 4** The diagram successively representing \( G''_P \) (cardinality operators for capacities), \( c_P \) (cardinality functions for capacities), \( \tau \) (functions defined on \( N_0 \), equal to 1 in 0), \( c \) (cardinality functions for bi-capacities) and \( G'' \) (cardinality operators for bi-capacities) sets, is commutative.

- In the foreground, we have the capacities whereas in the background, the bi-capacities are represented.
- On the left, the operators; on the right, the functions of a single variable.

- At the top, the triangular operators of \( S \) and \( S_P \); in the middle layer, the level operators and the level functions; at the bottom, the cardinality operators and the cardinality functions.

- The horizontal arrows correspond to group isomorphisms whereas the vertical ones stand for inclusions of sets (subgroups exactly).

6 **The inverse interaction transform**

We will now examine the automorphism \( inv \) on \( \tau \) defined by

\[
inv : \tau \rightarrow \tau,
\]

\[
f \mapsto f^{*-1}.
\]

Proposition 3.1. of [3] explicitely gives us the expression of \( f^{*-1} \). In particular, the inverse of the function \( f_{\gamma_P} \) is given by Proposition 3.3.:

\[
f_{\gamma_P}^{*-1}(0) = 1, \quad \text{and for } m \in N : 
\]

\[
f_{\gamma_P}^{*-1}(m) = -\frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k} f_{\gamma_P}^{*-1}(k).
\]

This last formula extended to natural numbers is named the Bernoulli sequence, which explains our former name inverse Bernoulli function for \( \gamma \) and \( \gamma_P \). The sequence \((b_m)_{m \in N} \) of Bernoulli numbers starts with

\[
1, -1/2, 1/6, 0, -1/30, \ldots
\]

and it is well known that \( b_m = 0 \) for \( m \geq 3 \) odd.
Yet, since it is the inversion in \( G \) that interests us, we use Proposition 4 which provides us the required inversion automorphism of \( G' \). On the other hand, since \( f_\gamma = f_{\gamma p} \), it is easy to find step by step, the inverse of \( \Gamma \):

\[
\Gamma \iff \gamma \mapsto f_\gamma = f_{\gamma p} \mapsto f_\gamma^{-1} = f_{\gamma p}^{-1} = (b_m)_{m \in \mathbb{N}_0} \mapsto \gamma^{-1} \mapsto \Gamma^{-1}
\]

This implies:

**Proposition 5** The Bernoulli operator \( \Gamma^{-1} \) is given by:

\[
\Gamma((A_1, B_1), (A_2, B_2)) = \begin{cases} 
    b_{b_1-b_2} & \text{if } (A_1, B_1) \preceq (A_2, B_2) \\
    0 & \text{otherwise}
\end{cases},
\]

\[(A_1, B_1), (A_2, B_2) \in \mathcal{Q}\]

where \((b_m)_{m \in \mathbb{N}}\) is the sequence of Bernoulli numbers.

As a consequence, thanks to the inversion of (7), we can write:

\[
m^v(A, B) = \sum_{(C,D) \in \mathcal{Q}(A,B)} b_{b-d} I^v(C, D) = \sum_{m=0} b_m \sum_{(C,D) \in \mathcal{Q}(A,B)} \sum_{b-d=m} I^v(C, D),
\]

\[(A, B) \in \mathcal{Q}.
\]

Finally, we obtain:

**Theorem 6** For any bi-capacity \( v \), we have:

\[
v(A, B) = \sum_{(C,D) \in \mathcal{Q}} b_{n-|D|} b_{n-|B\cup D\cup (A\cap C)|} I^v(C, D),
\]

\[(A, B) \in \mathcal{Q},
\]

where

\[
b^p_m := \sum_{k=0}^m \binom{m}{k} b_{p-k}
\]

for \( 0 \leq m \leq p \), and \((b_m)_{m \in \mathbb{N}}\) is the sequence of Bernoulli.

Let us notice that numbers \(b^p_m\) have been introduced in [3] to express a capacity \( v \) from its interaction index \( I^v \):

\[
v(A) = \sum_{C \in \mathcal{P}} b_{|C \cap A|} I^v(C), \ A \in \mathcal{P}.
\]

It is easy to compute them from the sequence of Bernoulli \((b^p_0 = b_p \text{ for any } p \in \mathbb{N})\), and thanks to the recursion of the “Pascal’s triangle”:

\[
b^p_{m+1} = b^p_m + b^p_{m-1}, \ 0 \leq m \leq p.
\]

Furthermore, the coefficients \(b^p_m\) show the following symmetry:

\[
b^p_m = (-1)^p b^p_{p-m}, \ 0 \leq m \leq p.
\]