Discussion on

Transversality of the Shapley value

By S. Moretti and F. Patrone

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The paper by S. Moretti and F. Patrone is a remarkable survey on the use of the Shapley value in many different domains (so different that one could also entitle the paper “Versatility of the Shapley value”). Although some other surveys on the Shapley value can be found, as indicated by the authors, they focus more on technical aspects (like axiomatization), while the present survey has a unique flavour, and moreover, is written in a very lively style which makes it a pleasure to read. I learned a lot reading these pages, and I discovered many domains very far from game theory where it was nevertheless possible to apply the Shapley value. It is a peculiarity of great fundamental concepts to be versatile, and also to be discovered many times in very different domains: one striking example given in this survey is that the pignistic transform proposed by Ph. Smets in the theory of belief functions (Sec. 11) is nothing else than the Shapley value. A similar and somehow related example worth to be mentionned is the Möbius transform (called dividends or unanimity coefficients in game theory, mass allocation in the theory of belief functions, multilinear form of pseudo-Boolean functions, etc.).

I am particularly grateful to the authors to have included a section on belief functions, a domain which is largely ignored by game theorists, despite the fact that it is mathematically very close to cooperative game theory: apart the appearance of the Shapley value and dividends, the core is also a central notion in this theory, and one can find theorems similar to the Shapley-Ichiishi result on convex games which have been derived independently, as well as the notion of selectope (see Chateauneuf and Jaffray [2]).

Another very interesting and original section is the one on applications to biology, which undoubtly opens a new door in a domain which is constantly developing. Equally interesting and original is the section on reliability theory.

I am also indebted to the authors for having mentionned the interaction index (in Section 9.2) which I proposed already about ten years ago, and which is a natural generalization (at least in the mathematical sense) of the Shapley value, since it quantifies the genuine contribution of a pair of players $i,j$ compared to the sum of contributions of $i$ and $j$. It is maybe worth mentionning that the first (up to my knowledge) apparition of the interaction index for pairs of players is due to Owen in 1972 [14], under the name of covalue, but it seems to have never been used by game theorists. It was rediscovered later by Murofushi and Soneda in 1993 [13], borrowing ideas from multiattribute utility theory. Then later, I generalized its definition to any coalition of players [6].

In what follows I would like to make some complementary remarks on several topics of this survey. The first ones concern extensions and generalizations of classical coalitional games.
The authors mention fuzzy games, multi-level (or multi-choice) games and so on. Another important class concerns games defined on ordered and/or combinatorial structures, such as matroids, convex geometries, lattices, regular set systems, and so on. There is a book by Bilbao [1] entirely devoted to this topic, gathering many of his personal works on the topic. I also wrote a recent survey for games defined on lattices [7], with emphasis on the Shapley value (see also [11]). Apart mathematical fun, the interest of these works is twofold: first, they can be used to define games where the actions a player can take are described in a more sophisticated way than the simple participation/non participation alternative. In its most general form, as I proposed in [11], each player has a partially ordered set of elementary actions at disposal, which can be combined in some way (this covers in particular multi-choice games). Secondly, we can describe a situation where not all coalitions are feasible. This is obviously the case if players are political parties, but many other examples arise in practice, and we could even say that this is more the rule than the exception. Pioneering works are due to Derks and Peters [3], and Faigle and Kern [4].

My second remark is still in the direction of generalization, and concerns coalition structures, a topic which is addressed in Section 5 of the present survey. The authors consider the case where the coalition structure (hence a partition of \( N \)) is fixed. But many works have been done for the so-called games in partition function form proposed by Thrall and Lucas [15], where games are defined for every partition \( P \) of \( N \) and every coalition \( S \) in \( P \). These games well depict situations where the worth of a coalition depends on externalities, however there are extremely complex to handle due to the highly combinatorial structure of embedded coalitions \((S, P)\). Just to give an idea, with 5 players there are 152 embedded coalitions, and with 8 players, 17,008 (compare respectively with 32 and 256, the number of coalitions). This explains why many attempts have been done to define a Shapley value for these games, but no consensus has been reached yet for a proper definition. For a recent work on this topic making a comparison with other attempts, see [12].

My third and last remark will concern applications. As well emphasized by the authors, the Shapley value is most often used in applications as a means to quantify the importance, power or relevance of some element in a problem. A large class of problems where this applies is the one of multicriteria decision problems and subjective evaluation. The aim of multicriteria decision making is to rank or score alternatives according to several (often conflicting) points of view, called criteria. This is done on the basis of the information given by the decision maker, who essentially gives his preference over a given set of typical alternatives. For example, alternatives could be apartments to rent, projects to select, candidates to hire, cars to buy, etc., while criteria are the rent, superficy, number of rooms, (for apartments), etc. Subjective evaluation is similar but concerns alternatives whose attributes are mainly measured by human experts: food, beverages, cosmetics, seats, etc. Here the decision maker is often a consumer, supposed to buy these products. A key concept here is to know whether a criterion is important or not to make decision. This is a particularly strategical issue when marketing new products. For reasons rooted in psychology, it is not possible to rely on the importance of criteria given directly by the decision maker, because he is unaware of his internal mechanism, if any, for making the final decision. The only reasonable approach is to infer the importance of criteria from the preference given by the decision maker. This can be done as follows. Consider \( N \) the set of criteria, and a capacity \( v : \mathcal{P}(N) \to [0, 1] \), where a capacity is a normalized monotone game, i.e., \( v(A) \leq v(B) \) if \( A \subseteq B \), and \( v(N) = 1 \). For any \( A \subseteq N \), \( v(A) \) is the overall score obtained by the alternative denoted by \( 1_A 0 \), where this notation means that for all criteria \( i \in A \), the level attained on the corresponding attribute is satisfactory (denoted by \( 1_i \)), while for the remaining criteria, the level attained is neutral (neither satisfactory nor unsatisfactory, denoted by \( 0_i \)). The capacity \( v \) can be obtained practically by elicitation of the preference of the decision maker on the set of
all possible alternatives $1_40$ using the MACBETH methodology (see [10]), or more generally by optimization methods using preference given on any set of alternatives (see [9] for a survey). Once $v$ obtained, the individual importance of criteria can be computed as the Shapley value of $v$. Moreover, the computation of the interaction index, at least for pairs of criteria, gives useful information on the way criteria interact for computing the overall score. Specifically, if the interaction between $i$ and $j$ is positive, it is necessary that both scores on $i$ and $j$ are high to get a high overall evaluation. If on the contrary the interaction is negative, then it suffices that one of the scores on $i$ and $j$ is high to get a high overall evaluation. The interested reader may find in [8] a detailed account of this methodology applied on the evaluation of discomfort of car seats. To finish this already long discussion, I would like to mention also that I have applied the above multicriteria approach to pattern recognition: there, the Shapley value can help to feature extraction, i.e., how to keep only relevant features for recognition [5].

References

