Bi-capacities

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Abstract—We introduce bi-capacities as a natural generalization of capacities (or fuzzy measures) through the identity of Choquet integral of binary alternatives with fuzzy measures. We examine the underlying structure and derive the Möbius transform of bi-capacities. Next, the Choquet and Sugeno integrals w.r.t. bi-capacities are introduced. It is shown that symmetric and asymmetric integrals are recovered. Lastly, we introduce the Shapley value and interaction indices. It is seen that besides a generalization based on the classical definitions, a definition involving two arguments is natural.

Keywords—bi-capacity, Choquet integral, Möbius transform, Shapley value, interaction index.

I. INTRODUCTION

Capacities [2] (or fuzzy measures [14], non-additive measures [3]), and integrals w.r.t capacities such as the Choquet integral [2] and the Sugeno integral [14], have become a central tool in decision making, extending e.g. expected utility models and linear models of multiattribute utility theory. While the so-called Choquet expected utility models are among the most general models for preference representation when utility functions are positive, the introduction of negative quantities for utility functions makes possible several extension of the usual Choquet expected utility model. Up to now, the most general extension is called the CPT model (Cumulative Prospect Theory [15]), and is a difference of two Choquet integrals.

More specifically, let us denote by $\nu_1, \nu_2$ two capacities on a finite universe $N$ of $n$ elements. For any real-valued function on $N$, the CPT model is expressed as:

$$\text{CPT}_{\nu_1, \nu_2}(f) := C_{\nu_1}(f^+) - C_{\nu_2}(f^-)$$

where $C_\nu$ is the Choquet integral with respect to the capacity $\nu$, and $f^+ := f \lor 0$, $f^- = (-f)^+$. We denote by $1_A$ the characteristic function of $A$, for any $A \subseteq N$. We call these functions binary functions, as they take only values 0 and 1. It is known that for any $A \subseteq N$, any capacity $\nu$, we have $C_\nu(1_A) = \nu(A)$. Hence the capacity is uniquely determined by giving the integral (or expected utility) of all binary functions. Let us consider ternary functions, i.e. functions valued on $\{-1, 0, 1\}$, which we express under the form $(1_A, -1_B)$, for $A, B \subseteq N$, $A \cap B = \emptyset$. Observe that

$$\text{CPT}_{\nu_1, \nu_2}(1_A, -1_B) = \nu_1(A) - \nu_2(B),$$

which entails that, if $\nu_1, \nu_2$ are given, we have no freedom for determining the utility of a ternary alternative (function), since this value is determined from the utility of two binary alternatives.

In order to get rid of this limitation, we introduce bi-capacities as the value assigned to a ternary function.

The paper presents first results on bi-capacities, their structure and machinery. We assume basic knowledge on capacities and Choquet integral (see e.g. [9] for background). To avoid heavy notations, we will often omit braces and commas to denote sets. Also, the cardinal of a set is denoted by the corresponding small letter, e.g. $|N| = n$.

II. BI-CAPACITIES

We denote $Q(N) := \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N)|A \cap B = \emptyset\}$. 

Definition 1: A function $v : Q(N) \rightarrow \mathbb{R}$ is a bi-capacity if it satisfies:

(i) $v(\emptyset, \emptyset) = 0$

(ii) $A \subseteq B$ implies $v(A, \cdot) \leq v(B, \cdot)$ and $v(\cdot, A) \geq v(\cdot, B)$. 
In addition, $v$ is normalized if $v(\emptyset, \emptyset) = 1 = -v(\emptyset, N)$.

The sequel, unless otherwise specified, we will consider that bi-capacities are normalized.

A bi-capacity is of the CPT type if there exists two (normalized) capacities $\nu_1, \nu_2$ such that

$$v(A, B) = v_1(A) - v_2(B), \forall (A, B) \in Q(N).$$

When $\nu_1 = \nu_2$, we say that the bi-capacity is symmetric, and asymmetric when $\nu_2 = \overline{\nu_1}$ (conjugate of $\nu_1$).

By analogy with the classical case, a bi-capacity is said to be additive if it satisfies for all $(A, B) \in Q(N)$:

$$v(A, B) = \sum_{i \in A} v(i, \emptyset) + \sum_{i \in B} v(\emptyset, i).$$

A generalization leads to the following definitions.

Definition 2: Let $S$ be a strict t-conorm. A bi-capacity is said to be $S$-decomposable if it satisfies:

$$v(A, B) = (S_{i \in A} v(i, \emptyset)) \ominus S (S_{i \in B} - v(\emptyset, i)))$$

where $\ominus$ is the $S$-difference (see [6]).

Definition 3: A bi-capacity is said to be max-decomposable or to be a bi-measure of possibility if

$$v(A, B) = \left[ \bigoplus_{i \in A} v(i, \emptyset) \right] \ominus \left[ \bigoplus_{i \in B} v(\emptyset, i) \right],$$

where $\bigoplus$ is the symmetric maximum [4] defined by

$$a \ominus b := \left\{ \begin{array}{ll} -(|a| \lor |b|) & \text{if } b \neq -a \text{ and either } |a| \lor |b| = -a \text{ or } = -b \\
0 & \text{if } b = -a \\
|a| \lor |b| & \text{else.} \end{array} \right.$$

Bi-capacities coincide with bicooperative games proposed by Bilbao et al. [1]. A similar concept has also been introduced by Greco et al. [11].
III. THE STRUCTURE OF $\mathcal{Q}(N)$

It is convenient to define a total order on $\mathcal{Q}(N)$, so as to reveal structures. A natural one is to use a ternary code, coding -1 by 0, 0 by 1, and 1 by 2. The increasing sequence of integers in ternary code is 0, 1, 2, 10, 11, 12, 20 etc., which leads to the following order of elements of $\mathcal{Q}(N)$:

$$
\cdots (2,3) (12,3) [([0,12) (0,2) (1,2) (0,1) (0,0) (1,0)] (2,1) (2,0) (12,0) (3,12) (3,2) \cdots
$$

We remark a fractal structure, which is enhanced by brackets.

When equipped with the following order: $(A, B) \leq (C, D)$ if $A \subset C$ and $B \supset D$, $(\mathcal{Q}(N), \leq)$ is the lattice $3^n$. Sup and inf are given by

$$(A, B) \lor (C, D) = (A \cup C, B \cap D),$$

$$(A, B) \land (C, D) = (A \cap C, B \cup D).$$

Top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. We give in figure 1 the Hasse diagram of $(\mathcal{Q}(N), \leq)$ for $n = 3$.

![Fig. 1. The lattice $\mathcal{Q}(N)$ for $n = 3$.](image)

**NOTE:** It is possible to have a more natural labelling of elements if we replace each element $(A, B)$ by $(A, B^*)$. Let us call $(\mathcal{Q}^*(N), \leq^*)$ this new lattice. An element $(A, B)$ in $\mathcal{Q}^*(N)$ is such that $A \subset B$. We have $(A, B) \leq^* (C, D)$ if and only if $A \subset C$ and $B \subset D$ (product order), and top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. All results in $\mathcal{Q}(N)$ have an equivalent form in $\mathcal{Q}^*(N)$. In the sequel, we stick to $\mathcal{Q}(N)$.

For any ordered pair $((A, B), (A \cup D, B \setminus C))$ of $\mathcal{Q}(N)$ with $C \subset B$ and $D \subset (N \setminus (A \cup B)) \cup C$ the interval $[(A, B), (A \cup D, B \setminus C)]$ is a sub-lattice of type $2^k \times 3^l$, with $k = |C \cap D|$, and $l = |C \cap D|$. As a particular case, a sub-lattice of type $2^k$ is obtained if $C \cap D = \emptyset$, and of type $3^l$ if $C = D$.

We use the notation

$$
\uparrow (A, B) := \{ (C, D) \in \mathcal{Q}(N) \mid (C, D) \geq (A, B) \}
$$

to denote the up-set of $(A, B)$.

As a last important remark, every element of $\mathcal{Q}(N)$ can be obtained as a supremum over a subset of $\{ (\emptyset, i^\emptyset), (i, i^\emptyset), \forall i \in N \}$ ($\lor$-irreducible elements). Specifically, for any $(A, B) \in \mathcal{Q}(N)$,

$$(A, B) = \bigvee_{i \in A} (i, i^\emptyset) \lor \bigvee_{j \in N \setminus B} (\emptyset, j^\emptyset).$$

In Fig. 1, $\lor$-irreducible elements are represented by black circles.

This permits to define layers in $\mathcal{Q}(N)$ as follows: $(\emptyset, N)$ is the bottom layer (layer 0), the set of all $\lor$-irreducible elements form layer 1, and layer $k$ for $k = 2, \ldots, n$, contains all elements which can be written as the supremum over exactly $k$ $\lor$-irreducible elements. Layer $k$ is denoted $\mathcal{Q}^k(N)$, and contains all elements $(A, B)$ such that $|B| = n - k$. The top layer (layer $n$) is reduced to $(N, \emptyset)$.

IV. MÖBIUS TRANSFORM OF BI-CAPACITIES

The general theory of Möbius transform starts from the Möbius function, which is an inversion operator over posets [13]. More specifically, we consider $f, g$ two real-valued functions on a locally finite poset $(X, \leq)$ such that

$$
g(x) = \sum_{y \leq x} f(y). \tag{2}
$$

The solution of this equation in term of $g$ is given through the Möbius function $\mu(x, y)$ by

$$
f(x) = \sum_{y \leq x} \mu(y, x) g(y) \tag{3}
$$

where $\mu$ is defined inductively and depends solely on $(X, \leq)$. Taking $(\mathcal{Q}(N), \leq)$ as a finite poset, it can be shown that the corresponding Möbius function is

$$
\mu((A, A'), (B, B')) =
\begin{cases}
(-1)^{|B \setminus A| + |A' \setminus B'|} & \text{if } (A, A') \leq (B, B') \text{ and } A' \cap B = \emptyset \\
0 & \text{otherwise}.
\end{cases}
$$

Consequently, the Möbius transform of $v$ is expressed by

$$
m(A, A') = \sum_{B \subseteq A, A' \subseteq B \subseteq A'} (-1)^{|B \setminus A| + |B' \setminus A'|} v(B, B'). \tag{4}
$$

By construction, we have

$$
v(A, A') = \sum_{(B, B') \leq (A, A')} m(B, B'). \tag{5}
$$

The following result is fundamental.
**Proposition 1:** Let \( v \) be a bi-capacity of the CPT type, with \( v(A, B) = v_1(A) - v_2(B) \). Then its Möbius transform is given by:

\[
m(A, A') = m^v(A), \forall A \subset N, A \neq \emptyset \\
m(\emptyset, B) = m^{\overline{v}}(B'), \forall B \subset N \\
m(\emptyset, N) = -1 \\
m(A, B) = 0 \text{ otherwise.}
\]

We get as immediate corollaries the expression of the Möbius transform of symmetric and asymmetric bi-capacities, and also of additive bi-capacities.

**Proposition 2:** Let \( v \) be an additive bi-capacity on \( Q(N) \). Then its Möbius transform is non null only for the \( v \)-irreducible elements and the bottom of \( Q(N) \). Specifically,

\[
m(\emptyset, i) = -v(\emptyset, i), \forall i \in N \\
m(i, i') = v(i, \emptyset), \forall i \in N \\
m(\emptyset, N) = -1.
\]

Proceeding as in [8], we may write the Möbius transform into a matrix form, using the total order we have defined on \( Q(N) \).

Denoting \( v, m \) put in vector form as \( v(n), m(n) \), Eq. (4) can be rewritten as

\[
m(n) = T(n) \circ v(n)
\]

where \( \circ \) is the usual matrix product, and \( T(n) \) is the matrix coding the Möbius transform. As with the case of classical capacities, \( T(n) \) has an interesting fractal structure, as it can be seen from the case \( n = 2 \) illustrated below.

\[
T(2) = \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

The generating element has the form

\[
\begin{bmatrix}
1 & -1 & -1 & 1
\end{bmatrix}
\]

and is the concatenation of two generating elements \([1, -1, 1, 1]\) of the Möbius transform for classical capacities [8].

The definition of the Möbius transform permits us to introduce \( k \)-additive bi-capacities.

**Definition 4:** A bi-capacity is said to be \( k \)-additive if its Möbius transform vanishes for all elements \((A, B) \) in \( Q([\emptyset]) \), for \( t = k + 1, \ldots, n \).

Equivalently, \( v \) is \( k \)-additive iff \( m(A, B) = 0 \) whenever \( |B| < n - k \). Clearly, \( 1 \)-additive bi-capacities coincide with additive bi-capacities.

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**V. THE CHOQUET AND SUGENO INTEGRALS W.R.T. BI-CAPACITIES**

The expression of the Choquet integral w.r.t a bi-capacity has been introduced axiomatically in [12], see also a presentation based on symmetry considerations in [7]. For any function \( f \) on \( N \), we denote by \( f_i \) the value \( f(i), i \in N \).

**Definition 5:** Let \( v \) be a bi-capacity and \( f \) be a real-valued function on \( N \). The Choquet integral of \( f \) w.r.t \( v \) is given by

\[
C_v(f) := C_{v_{N+}}(|f|)
\]

where \( v_{N+} \) is a real-valued set function on \( N \) defined by

\[
v_{N+}(C) := v(C \cap N^+, C \cap N^-),
\]

and \( N^+ := \{ i \in N | f_i \geq 0 \}, N^- := N \setminus N^+ \).

Observe that we have \( C_v(1_A, -1_B) = v(A, B) \) for any \((A, B) \in Q(N)\).

\( C_v(f) \) can be rewritten as:

\[
C_v(f) = \sum_{i=1}^{n} [f_{\sigma(i)}]\left[v(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-)
- v(A_{\sigma(i)+1} \cap N^+, A_{\sigma(i)+1} \cap N^-)\right]
\]

where \( A_{\sigma(i)} := \{ \sigma(i), \ldots, \sigma(n) \} \), and \( \sigma \) is a permutation on \( N \) so that \( |f_{\sigma(i)}| \leq \cdots \leq |f_{\sigma(n)}| \). The above formula is very similar to the one proposed by Greco et al. [11].

If \( v \) is of the CPT type, with \( v(A, B) = v_1(A) - v_2(B) \), the above formula reduces to

\[
C_v(f) = \sum_{i=1}^{n} [f_{\sigma(i)}]\left[v(A_{\sigma(i)} \cap N^+, A_{\sigma(i)} \cap N^-)
- v(A_{\sigma(i)+1} \cap N^+, A_{\sigma(i)+1} \cap N^-)\right]
- C_{v_1}(f^+) - C_{v_2}(f^-).
\]

Hence, our definition encompasses the CPT model.

A similar construction can be done with the Sugeno integral. Let \( S_v(f) \) denotes the Sugeno integral of \( f : N \rightarrow [0, 1] \) w.r.t. a capacity \( v \).

**Definition 6:** Let \( v \) be a bi-capacity and \( f : N \rightarrow [-1, 1] \). The Sugeno integral of \( f \) w.r.t \( v \) is defined by

\[
S_v(f) := S_{v_{N+}}(|f|)
\]

with the same notations as above.

However, since \( v_{N+} \) can assume negative values, the usual definition of Sugeno integral has to be replaced by the following one [5]

\[
S_v(f) := \langle \sum_{i=1}^{n} \left[ f_{\sigma(i)} \oplus v(A_{\sigma(i)}) \right] \rangle
\]

where \( f \) is assumed to be non negative, \( v \) is any real-valued set function such that \( v(\emptyset) = 0 \), and \( \sigma \) is a permutation on \( N \) such that \( f \) becomes non decreasing. Also, \( \oplus \) is the symmetric maximum, \( \odot \) the symmetric minimum defined by

\[
a \odot b := \left\{ \begin{array}{ll}
- |a| & \text{if} \ sign a \neq sign b \\
|a| & \text{else}
\end{array} \right.
\]

and for any sequence \( a_1, \ldots, a_n \) in \([-1, 1] \), the expression \( \langle \odot a_i \rangle \) is a shorthand for \( \langle \odot a_1 \rangle \odot (\cdots (\odot a_n) \odot a_i) \).
Then, the Sugeno integral for bi-capacities becomes
\[ S_v(f) = \left( \bigoplus_{i=1}^{n} \left[ f_{\sigma(i)} \otimes (A_{\sigma(i)} \cap N^+ \setminus N) \right] \right) \]  
(7)
where \( \sigma \) is a permutation on \( S \) such that \( |f| \) is non-decreasing.

Proposition 3: Let \( v \) be a bi-capacity satisfying \( v(A, B) = v_1(A) \oplus v_2(B) \) for all \( (A, B) \in Q(N) \), with \( v_1, v_2 \) being two normalized capacities on \( N \) \( (v \) is called a \( \vee \)-CPT capacity). Then the Sugeno integral reduces to
\[ S_v(f) := S_{v_2}(f^+) \oplus -S_{v_1}(f^-). \]

Note that if \( v_1 = v_2 \) \( (v \) could then be called a \( \vee \)-symmetric bi-capacity), then \( S_v \) is the symmetric Sugeno integral [5].

VI. DERIVATIVES OF BI-CAPACITIES

We extend the notion of derivative of a set function to bi-capacities (in fact to any function on \( Q(N) \)). As bi-capacities are defined on \( Q(N) \), so should be the variables used in derivation. For any \( i \in N \), the left-derivative with respect to \( i \) of \( v \) at point \( (S, T) \) is given by:
\[ \Delta_i v(S, T) := v(S \cup i, T) - v(S, T), \quad \forall (S, T) \in Q(N \setminus i). \]

Similarly, the right-derivative is given by:
\[ \Delta_i \overline{v}(S, T) := v(S, T \cup i) - v(S, T), \quad \forall (S, T) \in Q(N \setminus i). \]

The monotonicity of \( v \) entails that the left derivative is non negative, while the right derivative is non positive. One can also introduce the derivative w.r.t. \( i \) by
\[ \Delta_i v(S, T) := \Delta_i \overline{v}(S, T) - \Delta_i \overline{v}(S, T) = v(S \cup i, T) - v(S, T \cup i). \]

Left and right derivatives permit to define in general the derivative with respect to any element \((\emptyset, \emptyset) \neq (S, T) \) in \( Q(N) \) by the recursive relation:
\[ \Delta_{S,T} v(K, L) := \Delta_{i,0} (\Delta_{S \setminus i, T} v(K, L)) = \Delta_{\emptyset,1} (\Delta_{S, T \setminus \emptyset} v(K, L)), \]
for any \( (K, L) \in Q(N \setminus (S \cup T)) \). The general expression for the \((S, T)\)-derivative is given by:
\[ \Delta_{S,T} v(K, L) = \sum_{S' \subseteq S, T' \subseteq T} (-1)^{t(s-s')+(t-t')} v(K \cup S', L \cup T'), \]
for all \((K, L) \in Q(N \setminus (S \cup T)) \). As before, we introduce the derivative w.r.t. \( S \) for any \( S \subset N \)
\[ \Delta_S v(K, L) := \Delta_{S,0} v(K, L) = \Delta_{S,0} v(K, L). \]

We express the derivative in terms of the Möbius transform. The starting point is the following.

Lemma 1: For any \( i \in N \) and any \((S, T) \in Q(N \setminus i)\)
\[ \Delta_{i,0} v(S, T) = \sum_{(S', T') \in ((i, v'), (S, T))] m(S', T') \]
\[ \Delta_{0, i} v(S, T) = - \sum_{(S', T') \in ([0, v'), (S, T))] m(S', T') \]
\[ \Delta_i v(S, T) = \sum_{(S', T') \in ([0, v'), (S, T))] m(S', T'). \]

By induction, one can show the following general result.

Proposition 4: For any \((\emptyset, \emptyset) \neq (S, T) \) in \( Q(N) \),
\[ \Delta_{S,T} v(K, L) = (-1)^t \sum_{(S', T') \in [(\emptyset, v'), (S, T))] \bigvee_{j \in T} v_j, (S_i \setminus K, L) \]
\[ - (-1)^t \sum_{(S', T') \in [(\emptyset, v'), (K, L))] m(S', T'). \]

for any \((K, L) \in Q(N \setminus (S \cup T)) \) for first formula, and \( \forall (K, L) \in Q(N \setminus S) \) for the second one.

Remark that for any \((S, T) \in Q(N)\),
\[ \Delta_{S,T} v(\emptyset, N \setminus (S \cup T)) = m(S, N \setminus (S \cup T)) \]
\[ \sum_{(S, T) \in Q(N \setminus (S \cup T))} \Delta_{S,T} v(\emptyset, N \setminus (S \cup T)) \]
for any \((K, L) \in Q(N \setminus (S \cup T)) \) for first formula, and \( \forall (K, L) \in Q(N \setminus S) \) for the second one.

VII. BI-CAPACITIES AS BI-COOPERATIVE GAMES

We consider now bi-capacities as bi-cooperative games, as introduced by Bilbao et al. [1].

A. Unanimity games

A direct transposition of the notion of unanimity game leads to the following. Let \((S, S') \) in \( Q(N) \). The bi-unanimity game centered on \((S, S') \) is defined by:
\[ u_{(S,S')}(T, T') = \begin{cases} 1, & \text{if } T \supset S \text{ and } T' \subset S' \\ 0, & \text{otherwise} \end{cases} \]
(8)

It is easy to see by (5) that the Möbius transform of \( u_{(S,S')} \) is
\[ m^u_{(S,S')}(T, T') = \begin{cases} 1, & \text{if } (T', T') = (S, S') \\ 0, & \text{otherwise} \end{cases} \]

Hence, as in the classical case, the set of all bi-unanimity games is a basis for bi-capacities. Remark that \( u_{(S,S')} \) is not a normalized bi-capacity since \( u_{(S,S')} (\emptyset, N) \neq -1 \).

B. The Shapley value

The Shapley value for bi-cooperative games can be defined axiomatically by introducing axioms which are straightforward extensions of the original axioms, plus an additional symmetry axiom. For any player \( i \in N \), it can be shown that the Shapley value of \( i \) for \( v \) is:
\[ \phi(v)(i) = \sum_{S \subseteq N \setminus i} \frac{1}{n!} (n - s - 1)! \sum_{j \in S} \left[ v(S \cup \{i\}, N \setminus (S \cup i)) - v(S, N \setminus S) \right]. \]

Note that the term into brackets is simply \( \Delta_i v(S, N \setminus (S \cup i)) \).

It is immediate to see that if \( v \) is of the CPT type, i.e. \( v(S, T) = v_1(S) - v_2(T) \), then
\[ \phi(v)(i) = \phi^{v_1}(i) + \phi^{v_2}(i), \quad \forall i \in N, \]
where $\phi^{\nu_1}, \phi^{\nu_2}$ are the (classical) Shapley values of $\nu_1$ and $\nu_2$.

Recalling that for any game $v$, $\phi^{\nu}(i) = \phi^T(i)$ for any $i \in N$, we get $\phi^{\nu}(i) = 2\phi(i)$ for any symmetric or asymmetric game $v$.

If $v$ is an additive bi-capacity, we have $\phi^{\nu}(i) = v(i, \emptyset) - v(\emptyset, i)$.

Using Prop. 4, one can derive the expression of the Shapley value w.r.t. the Möbius transform

$$\phi^{\nu}(i) = \sum_{(S,S') \in \mathcal{E}(\emptyset, i^\nu)} \frac{1}{n - s'} m(S, S').$$ (10)

C. The interaction index

As in [10], the interaction index can be obtained from the Shapley value by a recursion formula. We first introduce necessary notions. Let $v$ be a bi-cooperative game on $N$, and let $\emptyset \neq K \subset N$. The restricted game $v^{N \setminus K}$ is the game $v$ restricted to players in $N \setminus K$, hence $v^{N \setminus K}(S, T) = v(S, T)$ for any $(S, T) \in Q(N \setminus K)$, and is not defined outside. The reduced game $v^K$ is the game where all players in $K$ are considered as a single player denoted $[K]$, i.e. the set of players is then $N[K] := (N \setminus K) \cup \{[K]\}$. The reduced game is defined by

$$v^K(S, T) := v([K], S) + v(T, [K])$$

for any $(S, T) \in Q(N[K])$, and $\phi([K]) : N[K] \rightarrow N$ is defined by

$$\phi([K]) := \begin{cases} S, & \text{if } [K] \not\subset S \\ (S \setminus [K]) \cup K, & \text{otherwise.} \end{cases}$$

Let us denote by $I^v(S)$ the interaction index for coalition $S \neq \emptyset$ in game $v$. The recursion formula is [10]

$$I^v(S) = I^v(S) - \sum_{K \subset S, K \neq \emptyset, S} I^{v^{N \setminus K}}(S \setminus K).$$

It can be shown that the interaction index writes

$$I^v(S) = \sum_{T \subset N \setminus S} \frac{(n - s - t)!}{(n - s + 1)!} \left[ \sum_{L \subset S} (-1)^{s-t} v(L \cup T, N \setminus (T \cup S)) - \sum_{L \subset S} (-1)^{s-t} v(T, N \setminus (L \cup T)) \right].$$

Observe that this may be written also

$$I^v(S) = \sum_{T \subset N \setminus S} \frac{(n - s - t)!}{(n - s + 1)!} \Delta_{S^T}(v(T, N \setminus (S \cup T))).$$

It can be shown that the expression w.r.t. the Möbius transform is

$$I^v(S) = \sum_{(S', T') \geq (S, T)} \frac{1}{n - t' - s + 1} m(S', T') + (-1)^{s-t} \sum_{(S', T') \in Q(N \setminus S)} \frac{1}{n - t' - s + 1} m(S', T').$$

This expression shows that if $v$ is $k$-additive, then $I^v(S) = 0$ for any $S$ of more than $k$ elements, and for any $S$ of exactly $k$ elements, $I^v(S) = m(S, N \setminus S) + (-1)^{s-t} m(\emptyset, N \setminus S)$. If $v$ is of the CPT type, with $v(S, T) = \nu_1(S) - \nu_2(T)$, then

$$I^v(S) = I^{\nu_1}(S) + (-1)^{s-t} I^{\nu_2}(S),$$

where $I^{\nu_1}$ is the (classical) interaction index of $\nu_1$. Since $I^{\nu_2}(S) = (-1)^{s-t} I^v(S)$, we have $I^v(S) = 0$ when $s$ is even and $v$ is asymmetric.

D. The bi-interaction index

Since bi-cooperative games are defined on $Q(N)$, the interaction index should be defined for all coalitions in $Q(N)$. The most natural definition seems to use the derivative, as for the classical case. We propose the following

Definition 7: Let $(S, T) \in Q(N)$. The bi-interaction index w.r.t $(S, T)$ is defined by

$$I^v(S, T) := \sum_{K \subset N \setminus (S \cup T)} \frac{(n - s - t - k)!}{(n - s - t + 1)!} \Delta_{S^T}(v(K, N \setminus (K \cup S \cup T))).$$

A first observation is that for any $S \subset N$

$$I^v(S) = I^v(S, \emptyset) - I^v(\emptyset, S).$$

Proposition 5: Let $v$ be a bi-cooperative game on $N$. For any $(S, T) \subset Q(N)$,

$$I^v(S) = \frac{(-1)^t}{(n - s - t - t')!} \sum_{(S', T') \in \{(S, T) \setminus \emptyset, \emptyset \} \cap Q(N \setminus T)} \frac{1}{n - s - t - t' + 1} m(S', T').$$

Lastly, we examine the case of $k$-additive bi-capacities and CPT-type bi-capacities.

Proposition 6: (i) If $v$ is a $k$-additive bi-capacity, then

(i.1) $I^v(S, T) = 0$, for all $(S, T) \in Q(N)$ such that $|S \cup T| > k$.

(ii.2) $I^v(S, T) = (-1)^s m(S, N \setminus (S \cup T))$, for all $(S, T) \in Q(N)$ such that $|S \cup T| = k$.

(ii) If $v$ is of CPT type, then $I^v(S, T) = 0$ unless $S = \emptyset$ or $T = \emptyset$.

REFERENCES


